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Cyril LEVY

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Moulay-Tahar Benameur, Professeur à l'Université Paul Verlaine, Metz

Ludwik Dąbrowski, Professeur à SISSA, Trieste (Rapporteur)

Bruno Iochum, Professeur à l'Université de Provence, Marseille (Directeur de thèse)

Sylvie Paycha, Professeur à l'Université Blaise Pascal, Clermont-Ferrand (Présidente, Rapporteur)

Michael Puschnigg, Professeur à l'Université de la Méditerranée, Marseille

Thomas Schücker, Professeur à l'Université de Provence, Marseille

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Introduction

This thesis is a presentation of the research I have done during the last three years in collaboration with Driss Essouabri, Bruno Iochum and Andrzej Sitarz.

Noncommutative geometry is a vast field of mathematics which aims at extending classical geometry in a noncommutative setting. Specifically, using functional analysis, operator algebras, spectral theory and spin geometry, noncommutative geometry extends for instance concepts of locally compact topological space, and spin Riemannian manifold.

The interest for this mathematical field goes beyond the purely mathematical. There are profound physical motivations behind the concepts of noncommutative geometry, which suggest that it can be used to describe the basic elements of physics such as spacetime and quantum fields. More specifically, noncommutative geometry appears, as a mathematical framework, well-suited to a geometric formulation of quantum concepts.

One may consider that noncommutative geometry was born with the following theorem of Gelfand–Naimark: any commutative C^* -algebra is isomorphic to the C^* -algebra of continuous functions on a compact space, namely the space of characters of the algebra. Since all the topological information of a space is contained in the set of continuous functions on the space, we can see that the notion of C^* -algebra is a generalization of the notion of compact topological space.

From this fundamental result, it has been possible to go beyond purely topological concepts and construct a full noncommutative Riemannian differential geometry, with its own version of differential and integral calculus, fiber bundles, measures, spin manifolds, etc... This colossal work has been pioneered and essentially done by Alain Connes [26–30].

Noncommutative geometry makes us change our point of view: it is not the set of points (the topological space) that is fundamental, but the set of functions (defined on the set of points). While equivalent in the commutative world, these two points of view are unequivalent in the noncommutative world.

Noncommutativity and this change of point of view are two crucial characteristics of the fundamental structure of quantum physics. Indeed, in quantum physics, observables do not always commute, contrarily to the case of classical physics. Moreover, $f(x)$, namely the evaluation of an observable f at a point x , is not defined. However, the notion of observable still exists and $x(f)$ has a meaning, provided that x is a state, a noncommutative equivalent of the notion of character of an algebra that can evaluate the observables f .

This striking similarity between noncommutative geometry and quantum theory is very important and constitutes a source of the development of noncommutative geometry and its applications in physics. In fact, any C^* -algebra (commutative or not) is isomorphic to a closed

subalgebra of the algebra of bounded operators on a Hilbert space. This shows that noncommutative geometry is well suited to the formalism of quantum physics.

Noncommutative geometry provides new mathematical notions that can be used to extend the fundamental aspect of noncommutativity of observables to spacetime itself. The two theories of fundamental interactions, namely quantum field theory (standard model) for the strong and electroweak interactions and general relativity for the gravitational interaction, are not based on the same mathematical formalism. The first one is based on a quantum point of view, while the latter one is classical. Moreover, spacetime itself is not considered the same way by these two theories. In quantum field theory, spacetime is fixed (Minkowskian flat spacetime) while in general relativity, spacetime is dynamical and gravitational effects are consequences of the curvature of the spacetime metric. These fundamental differences are not problematic if we study a physical phenomenon for which gravitational interaction can be neglected or inversely for which quantum effects can be neglected. Indeed, both theories are very powerful in their respective domain of application.

However, for the study of phenomena that manifestly involve all known interactions (compact objects, black holes, big-bang), it is necessary to make these theories compatible, and to reunite them under a unique coherent mathematical formalism. The main idea that has been considered by theoretical physicists is to develop a generalization of quantum field theory in order to incorporate gravitation, or in other words, to quantize gravity.

The pursuit of this goal has been realized through various approaches. One of these approaches is string theory, which is based on an increase in the number of dimensions, certain dimensions being compactified, and another approach is loop quantum gravity, which uses a spin foam structure for spacetime without using a background spacetime metric, contrarily to string theory. None of these theories have been confirmed by experiments yet, and theoretical predictions are difficult to obtain.

The approach suggested by noncommutative geometry is based on a noncommutative generalization of the Lorentzian manifold which represents spacetime. By introducing noncommutativity at the spacetime level, this approach allows to apprehend the impossibility of spacetime continuity, as suggested by quantum mechanics, and the intrinsic limit associated to Planck length $l_p = \sqrt{\frac{\hbar G}{c^3}} \approx 10^{-33}$ cm. Actually, this approach successfully unified, at the classical level, the three interactions of the standard model with gravitation, and provided a geometric interpretation of the Higgs mechanism in particle physics. The fundamental object at the interface between noncommutative geometry and quantum field theory is the notion of spectral triple, which is a noncommutative generalization of the notion of spin Riemannian manifold, the starting point for the construction of physical theories. By considering an “almost commutative” spectral triple, that is to say the product of a commutative spectral triple (a compact Riemannian spin manifold, describing the “continuous” spacetime), by a zero dimensional spectral triple (based on a matricial algebra), it is possible to simultaneously recover the standard model and general relativity. The fundamental ingredient in this unifying theory is the Chamseddine–Connes spectral action $S = \text{Tr } \Phi(\mathcal{D}/\Lambda)$, which is an action functional defined on the preceding almost commutative spectral triple and which is defined through the spectrum of the Dirac operator \mathcal{D} associated to this spectral triple, where Φ is a cut-off function and Λ is a mass scale parameter. Specifically, the spectral action unifies the electroweak, strong and gravitational interactions [17, 21–25, 33, 37, 86, 87, 131, 136], and corresponds to the number of eigenvalues of the Dirac operator that are less or equal to a given mass scale Λ . By proceeding to a fluctuation

of the metric, that is to say, a generalized gauge transformation associated to the unitary elements of the spectral triple algebra, it is possible, by expanding the spectral action in powers of Λ , to recover the standard model Lagrangian, coupled to the Einstein–Hilbert gravitational Lagrangian.

Another very important question associated to applications in physics of noncommutative geometry concepts concerns the renormalization in quantum field theory. Renormalization is a fundamental mathematical process that extracts physical information (finite numerical quantities) from mathematical expressions corresponding to infinite quantities. It has been shown [35,95] that the combinatorics of the perturbative renormalization in quantum field theory are related to the theory of Hopf algebras, which are the noncommutative Lie groups of symmetry in the language of noncommutative geometry. Moreover, Dixmier trace and Wodzicki residue, which are related to renormalization, have been extended by Connes [28,29] in the noncommutative setting of spectral triples. Thus, we expect that conceptual aspects of renormalization can be incorporated in a noncommutative formalism of quantum field theory.

In this thesis, we discussed certain mathematical issues related to the computation of the spectral action on some fundamental noncommutative spectral triples, such as the noncommutative torus and the quantum 3-sphere $SU_q(2)$. We also studied the question of existence of tadpoles (linear terms in the potential A of the fluctuation of the metric in the spectral action) in the case of commutative Riemannian geometries, and the construction of a symbolic global pseudodifferential calculus allowing a generalization of the Weyl–Moyal product on a Schwartz space of rapidly decaying sections on a cotangent bundle of a manifold with linearization.

This text is divided into 5 chapters.

The first and second chapters present the work done in collaboration with Driss Essouabri, Bruno Iochum and Andrzej Sitarz in the paper *Spectral action on noncommutative torus* [53].

The first chapter is a review of definitions and properties about spectral triples, pseudodifferential calculus and spectral action. We establish in section 1.3 some results about residues of zeta functions ζ_{D_A} that will be used subsequently in chapter 2, 3. We also see some properties about linear terms in spectral actions (tadpoles) which will be used in chapter 4.

We compute, in the second chapter, the spectral action on noncommutative torus through heat kernel expansion and residues of Hurwitz–E Epstein zeta functions. The computation is based on expansions of terms $f|D_A|^{-k}$, $\zeta_{D_A}(0)$, as noncommutative integrals of certain pseudodifferential operators. We applied these results to the n -noncommutative torus $(C^\infty(\mathcal{T}_\Theta^n), \mathcal{H}, \mathcal{D})$, which is a simple spectral triple. The full spectral action obtained on the noncommutative torus is, in dimension 4,

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = 8\pi^2 \Phi_4 \Lambda^4 - \frac{4\pi^2}{3} \Phi(0) \tau(F_{\mu\nu} F^{\mu\nu}) + \mathcal{O}(\Lambda^{-2}),$$

which shows that a new noncommutative Yang–Mills term $\tau(F_{\mu\nu} F^{\mu\nu})$, where $F_{\mu\nu} := \delta_\mu(A_\nu) - \delta_\nu(A_\mu) - [A_\mu, A_\nu]$, replaces the standard one. For the computation, we had to investigate holomorphic continuation of certain series of zeta functions. In particular, we saw that a Diophantine constraint on the Θ matrix of deformation naturally appears as crucial in the computation of the spectral action, when the reality operator J is taken into account. In other words, a Diophantine condition on Θ appears when the perturbation $D \rightarrow D + A + \varepsilon J A J^{-1}$ is considered, but is not needed for a perturbation of the type $D \rightarrow D + A$, where A is a selfadjoint one form.

The third chapter presents the work done in collaboration with Bruno Iochum and Andrzej Sitarz in the paper *Spectral action on $SU_q(2)$* [82]. The spectral triple based on the $SU_q(2)$

quantum group can be seen as q -deformed noncommutative 3-sphere [48]. We compute the full spectral action on $SU_q(2)$ by taking into account the real structure J . The dimension spectrum of $SU_q(2)$ being a finite set, there is only a finite number of terms in the spectral action expansion. Moreover, it appears that tadpoles exist on $SU_q(2)$. We show that the action depends on q and the limit $q \rightarrow 1$ does not exist automatically. When it exists, such limit does not lead to the associated action on the commutative sphere \mathbb{S}^3 . On $SU_q(2)$, we show that the sign F of the Dirac operator has very special properties: first, it commutes modulo $OP^{-\infty}$ with the algebra, and second, it can be seen as a selfadjoint one-form, giving terms which are independent of q in the spectral action. In order to obtain the spectral action, we used the previous pseudodifferential calculus defined in chapter 1 and a Poincaré–Birkhoff–Witt decomposition of 1-forms. It appears that a very special multiplicative behavior of noncommutative integrals with the real structure J holds on $SU_q(2)$: indeed, we show in section 2.4.6 that

$$\oint A J B J^{-1} |\mathcal{D}|^{-3} = \frac{1}{2} \oint A |\mathcal{D}|^{-3} \overline{\oint B |\mathcal{D}|^{-3}}$$

where A and B are δ -1-forms (linear combinations of terms of the form $a[|\mathcal{D}|, b]$ with $a, b \in \mathcal{A}$). These results prove that the spectral action on $SU_q(2)$ is completely determined by the terms

$$\oint A^q |\mathcal{D}|^{-p}, \quad 1 \leq q \leq p \leq 3,$$

where A is a δ -1-form. We then tackle the precise computation of these noncommutative integrals. In order to realize these computations, we establish a differential calculus up to some ideal \mathcal{R} in pseudodifferential operators.

The fourth chapter presents the work done in collaboration with Bruno Iochum in the paper *Tadpoles and commutative spectral triples* [83]. In quantum field theory, a tadpole is a one-loop Feynman diagram with one external line, giving a contribution to the vacuum expectation value of the field. In the setting of noncommutative geometry, these diagrams are represented by linear terms in the Chamseddine–Connes spectral action. We show in this chapter that there are no tadpoles of any order in the spectral action of compact spin manifolds without boundary, and no tadpoles of order less than 5 on manifolds with boundary with chiral boundary condition. Using pseudodifferential techniques and Wodzicki residue, we track zero terms in spectral actions of compact spin manifolds. We expect that most of these results can be extended to manifolds with boundary under very general boundary conditions.

The fifth chapter presents a work [101] about global pseudodifferential calculus on manifold with linearization. In [59], Gayral et al. have established a remarkable link between deformation quantization and Connes’ noncommutative geometry. It has been proven that Moyal planes are (noncompact) spectral triples. The Moyal product is a star-product defined on the Schwartz space $\mathcal{S}(\mathbb{R}^{2n})$ and yields a Fréchet pre- C^* -algebra structure on that space. The noncompact spectral triple described in [59] was built on this algebra.

We propose in this chapter the construction of a global pseudodifferential calculus that allows to extend the construction of the Moyal product to more general spaces, the main goal being the construction of new noncommutative spectral triples based on deformation quantization star-products. Specifically, the Moyal product that we obtain is defined on the Schwartz space of rapidly decaying functions on the cotangent bundle of a manifold. It corresponds to the transfer

in the symbol space of the kernel convolution product through a quantization isomorphism. We consider the case of manifolds M with linearization in the sense of Bokobza-Haggiag [9], such that the associated (abstract) exponential map provides global diffeomorphisms of M with \mathbb{R}^n at any point. Cartan–Hadamard manifolds are special cases of such manifolds. The abstract exponential map encodes a notion of infinity on the manifold that allows, modulo some hypothesis of S_σ -bounded geometry, to define the Schwartz space of rapidly decaying functions, globally defined Fourier transformation and classes of symbols with uniform and decaying control over the x variable. Given a linearization on the manifold with some properties of control at infinity, we construct symbol maps and λ -quantization, explicit Moyal star-product on the cotangent bundle, and classes of pseudodifferential operators. We show that these classes are stable under composition, and that the λ -quantization map gives an algebra isomorphism (which depends on the linearization) between symbols and pseudodifferential operators. We study L^2 -continuity and give some examples. We show in particular that the hyperbolic 2-space \mathbb{H} has a S_1 -bounded geometry, allowing the construction of a global symbol calculus of pseudodifferential operators, and an intrinsic Moyal product on $\mathcal{S}(\mathbb{H})$.

A summary in French of this thesis is given in appendix.

Chapitre 1

Spectral action on spectral triples

1.1 Introduction

The spectral action introduced by Chamseddine–Connes [21] plays an important role in non-commutative geometry. More precisely, given a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ where \mathcal{A} is an algebra acting on the Hilbert space \mathcal{H} and \mathcal{D} is a Dirac-like operator (see [28, 68]), they proposed a physical action depending only on the spectrum of the covariant Dirac operator

$$\mathcal{D}_A := \mathcal{D} + A + \epsilon J A J^{-1} \quad (1.1)$$

where A is a one-form represented on \mathcal{H} , so has the decomposition

$$A = \sum_i a_i [\mathcal{D}, b_i], \quad (1.2)$$

with $a_i, b_i \in \mathcal{A}$, J is a real structure on the triple corresponding to charge conjugation and $\epsilon \in \{1, -1\}$ depending on the dimension of this triple and comes from the commutation relation

$$J\mathcal{D} = \epsilon \mathcal{D}J. \quad (1.3)$$

In this chapter, we revisit the notions of pseudodifferential operators on spectral triples, zeta functions, noncommutative integral, dimension spectrum and spectral action. The reality operator J is incorporated and we pay a particular attention to kernels of operators which can play a role in the constant (scale invariant) term of the spectral action.

1.2 Noncommutative integration on a simple spectral triple

1.2.1 Pseudodifferential operators on spectral triples

Noncommutative geometry, with its notion of spectral triple, provides a minimal data which allows to start doing quantum field theory.

Definition 1.2.1. A triplet $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is called a spectral triple if \mathcal{A} is a unital $*$ -algebra faithfully represented as bounded operators on the Hilbert space \mathcal{H} , and \mathcal{D} is a selfadjoint operator with compact resolvent such that all commutators $[\mathcal{D}, a]$ are bounded for $a \in \mathcal{A}$.

A spectral triple is finitely summable, with dimension n , if the resolvent set of \mathcal{D} has characteristic values $\lambda_j = \mathcal{O}(j^{-1/n})$ when $j \rightarrow \infty$.

A spectral triple is said regular if \mathcal{A} and $[\mathcal{D}, \mathcal{A}]$ are in $\cap_{k \in \mathbb{N}} \text{Dom } \delta^k$ where $\delta(T) := [[\mathcal{D}], T]$.

If n is even, we shall suppose that there is a $\mathbb{Z}/2$ -grading χ (the chirality operator) which commutes with any element of \mathcal{A} and anticommutes with \mathcal{D} . In this formalism, \mathcal{D} plays the role of the Dirac operator in classical Riemannian spin geometry. In other words, \mathcal{D}^{-1} will represent the (Euclidean) fermion propagator. In order to have a charge conjugation in our theory, one adds a real structure, which brings an antiunitary operator J which commutes or anticommutes with \mathcal{D} . In this setting, the gauge bosons will be seen as inner fluctuation of the Dirac operator $\mathcal{D} \rightarrow \mathcal{D}_A := \mathcal{D} + A + \varepsilon J A J^{-1}$, where A is a selfadjoint 1-form, which is an operator of the form $\sum a_i [\mathcal{D}, b_i]$, where a_i and b_i are in \mathcal{A} .

The notion of real structure on a finite summable spectral triple is related to real K -homology, and is defined by:

Definition 1.2.2. A real structure J on a finitely summable spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of dimension n is an antilinear isometry J such that $J\mathcal{D} = \varepsilon \mathcal{D}J$, $J^2 = \varepsilon'$, $J\chi = \varepsilon''\chi J$ (even case), where $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$ and satisfy the following table

$n \bmod 8$	0	1	2	3	4	5	6	7
$J^2 = \pm 1$	+	+	-	-	-	-	+	+
$J\mathcal{D} = \pm \mathcal{D}J$	+	-	+	+	+	-	+	+
$J\chi = \pm \chi J$	+		-		+		-	

Moreover, any element of $J\mathcal{A}J^{-1}$ commutes with any element of $\mathcal{A} \cup [\mathcal{D}, \mathcal{A}]$.

A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ endowed with a real structure J is called a real spectral triple.

Remark that the notion of spectral triple can actually be extended to the framework of von Neumann algebras [6].

For a given 1-form A on a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, we will have to compare here the kernels of \mathcal{D} and \mathcal{D}_A . Note first that they are finite dimensional:

Lemma 1.2.3. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple with a reality operator J and chirality χ . If $A \in \Omega_{\mathcal{D}}^1$ is a one-form, the fluctuated Dirac operator*

$$\mathcal{D}_A := \mathcal{D} + A + \varepsilon J A J^{-1}$$

is an operator with compact resolvent, and in particular its kernel $\text{Ker } \mathcal{D}_A$ is a finite dimensional space. This space is invariant by J and χ .

Proof. Let T be a bounded operator and let z be in the resolvent set of $\mathcal{D} + T$ and z' be in the resolvent set of \mathcal{D} . Then

$$(\mathcal{D} + T - z)^{-1} = (\mathcal{D} - z')^{-1} [1 - (T + z' - z)(\mathcal{D} + T - z)^{-1}].$$

Since $(\mathcal{D} - z')^{-1}$ is compact by hypothesis and since the term in bracket is bounded, $\mathcal{D} + T$ has compact resolvent. Applying this to $T = A + \varepsilon J A J^{-1}$, \mathcal{D}_A has a finite dimensional kernel (see for instance [91, Theorem 6.29]).

Since according to the dimension, $J^2 = \pm 1$, J commutes or anticommutes with χ , χ commutes with the elements in the algebra \mathcal{A} and $\mathcal{D}\chi = -\chi\mathcal{D}$ (see [30] or [68, p. 405]), we get $\mathcal{D}_A\chi = -\chi\mathcal{D}_A$ and $\mathcal{D}_AJ = \pm J\mathcal{D}_A$ which gives the result. \square

We now set $(\mathcal{A}, \mathcal{D}, \mathcal{H}, J)$ a given real regular spectral triple of dimension n and A a selfadjoint one-form. We denote

$$\begin{aligned} P_0 & \text{ the projection on } \text{Ker } \mathcal{D}, P_A \text{ the projection on } \text{Ker } \mathcal{D}_A, \\ D & := \mathcal{D} + P_0, D_A := \mathcal{D}_A + P_A. \end{aligned}$$

P_0 and P_A are thus finite-rank selfadjoint bounded operators. Remark that D and D_A are selfadjoint invertible operators with compact inverses.

Remark 1.2.4. *Since we only need to compute the residues and the value at 0 of the ζ_D, ζ_{D_A} functions, it is not necessary to define the operators \mathcal{D}^{-1} or \mathcal{D}_A^{-1} and the associated zeta functions. However, we can remark that all the work presented here could be done using the process of Higson in [79] which proves that we can add any smoothing operator to \mathcal{D} or \mathcal{D}_A such that the result is invertible without changing anything to the computation of residues (see also [37], where this question is considered).*

Define for any $\alpha \in \mathbb{R}$

$$\begin{aligned} OP^0 & := \{T : t \mapsto F_t(T) \in C^\infty(\mathbb{R}, \mathcal{B}(\mathcal{H}))\}, \\ OP^\alpha & := \{T : T|D|^{-\alpha} \in OP^0\}. \end{aligned}$$

where $F_t(T) := e^{it|D|} T e^{-it|D|} = e^{it|D|} T e^{-it|D|}$ since $|D| = |\mathcal{D}| + P_0$. Define

$$\begin{aligned} \delta(T) & := [|D|, T], \\ \nabla(T) & := [\mathcal{D}^2, T], \\ \sigma_s(T) & := |D|^s T |D|^{-s}, s \in \mathbb{C}. \end{aligned}$$

It has been shown in [38] that $OP^0 = \bigcap_{p \geq 0} \text{Dom}(\delta^p)$. In particular, OP^0 is a subalgebra of $\mathcal{B}(\mathcal{H})$ (while elements of OP^α are not necessarily bounded for $\alpha > 0$) and $\mathcal{A} \subseteq OP^0$, $JAJ^{-1} \subseteq OP^0$, $[\mathcal{D}, \mathcal{A}] \subseteq OP^0$. Note that $P_0 \in OP^{-\infty}$ and $\delta(OP^0) \subseteq OP^0$.

For any $t > 0$, \mathcal{D}^t and $|\mathcal{D}|^t$ are in OP^t and for any $\alpha \in \mathbb{R}$, D^α and $|D|^\alpha$ are in OP^α . By hypothesis, $|D|^{-n} \in \mathcal{L}^{(1, \infty)}(\mathcal{H})$ so for any $\alpha > n$, $OP^{-\alpha} \subseteq \mathcal{L}^1(\mathcal{H})$.

Lemma 1.2.5. [38]

- (i) For any $T \in OP^0$ and $s \in \mathbb{C}$, $\sigma_s(T) \in OP^0$.
- (ii) For any $\alpha, \beta \in \mathbb{R}$, $OP^\alpha OP^\beta \subseteq OP^{\alpha+\beta}$.
- (iii) If $\alpha \leq \beta$, $OP^\alpha \subseteq OP^\beta$.
- (iv) For any α , $\delta(OP^\alpha) \subseteq OP^\alpha$.
- (v) For any α and $T \in OP^\alpha$, $\nabla(T) \in OP^{\alpha+1}$.

Proof. (i) We have $|D|T|D|^{-1} = T + \delta(T)|D|^{-1}$ and $|D|^{-1}T|D| = T - |D|^{-1}\delta(T)$. A recurrence proves that for any $k \in \mathbb{N}$, $|D|^k T |D|^{-k} = \sum_{q=0}^k \binom{k}{q} \delta^q(T) |D|^{-q}$ and we get $|D|^{-k} T |D|^k = \sum_{q=0}^k (-1)^q \binom{k}{q} |D|^{-q} \delta^q(T)$.

As a consequence, since T , $|D|^{-q}$ and $\delta^q(T)$ are in OP^0 for any $q \in \mathbb{N}$, for any $k \in \mathbb{Z}$, $|D|^k T |D|^{-k} \in OP^0$. Let us fix $p \in \mathbb{N}_0$ and define $F_p(s) := \delta^p(|D|^s T |D|^{-s})$ for $s \in \mathbb{C}$. Since for $k \in \mathbb{Z}$, $F_p(k)$ is bounded, a complex interpolation proves that $F_p(s)$ is bounded, which gives $|D|^s T |D|^{-s} \in OP^0$.

- (ii) Let $T \in OP^\alpha$ and $T' \in OP^\beta$. Thus, $T|D|^{-\alpha}$, $T'|D|^{-\beta}$ are in OP^0 . By (i) we get $|D|^\beta T|D|^{-\alpha}|D|^{-\beta} \in OP^0$, so $T'|D|^{-\beta}|D|^\beta T|D|^{-\beta-\alpha} \in OP^0$. Thus, $T'T|D|^{-(\alpha+\beta)} \in OP^0$.
- (iii) For $T \in OP^\alpha$, $|D|^{\alpha-\beta}$ and $T|D|^{-\alpha}$ are in OP^0 , thus $T|D|^{-\beta} = T|D|^{-\alpha}|D|^{\alpha-\beta} \in OP^0$.
- (iv) follows from $\delta(OP^0) \subseteq OP^0$.
- (v) Since $\nabla(T) = \delta(T)|D| + |D|\delta(T) - [P_0, T]$, the result follows from (ii), (iv) and the fact that P_0 is in $OP^{-\infty}$. \square

Remark 1.2.6. Any operator in OP^α , where $\alpha \in \mathbb{R}$, extends as a continuous linear operator from $\text{Dom } |D|^{\alpha+1}$ to $\text{Dom } |D|$ where the $\text{Dom } |D|^\alpha$ spaces have their natural norms (see [38, 79]).

We now introduce a definition of pseudodifferential operators in a slightly different way than in [29, 38, 79] which in particular pays attention to the reality operator J and the kernel of \mathcal{D} and allows \mathcal{D} and $|D|^{-1}$ to be pseudodifferential operators. It is more in the spirit of [22].

Definition 1.2.7. Let us define $\mathcal{D}(\mathcal{A})$ as the polynomial algebra generated by \mathcal{A} , JAJ^{-1} , \mathcal{D} and $|\mathcal{D}|$.

A pseudodifferential operator is an operator T such that there exists $d \in \mathbb{Z}$ such that for any $N \in \mathbb{N}$, there exist $p \in \mathbb{N}_0$, $P \in \mathcal{D}(\mathcal{A})$ and $R \in OP^{-N}$ (p , P and R may depend on N) such that $PD^{-2p} \in OP^d$ and

$$T = PD^{-2p} + R.$$

Define $\Psi(\mathcal{A})$ as the set of pseudodifferential operators and $\Psi(\mathcal{A})^k := \Psi(\mathcal{A}) \cap OP^k$.

Note that if A is a 1-form, A and JAJ^{-1} are in $\mathcal{D}(\mathcal{A})$ and moreover $\mathcal{D}(\mathcal{A}) \subseteq \cup_{p \in \mathbb{N}_0} OP^p$. Since $|\mathcal{D}| \in \mathcal{D}(\mathcal{A})$ by construction and P_0 is a pseudodifferential operator, for any $p \in \mathbb{Z}$, $|D|^p$ is a pseudodifferential operator (in OP^p). Let us remark also that $\mathcal{D}(\mathcal{A}) \subseteq \Psi(\mathcal{A}) \subseteq \cup_{k \in \mathbb{Z}} OP^k$.

Lemma 1.2.8. [29, 38] The set of all pseudodifferential operators $\Psi(\mathcal{A})$ is an algebra. Moreover, if $T \in \Psi(\mathcal{A})^d$ and $T' \in \Psi(\mathcal{A})^{d'}$, then $TT' \in \Psi(\mathcal{A})^{d+d'}$.

Proof. The non-trivial part of the proof is the stability under the product of operators. Let $T, T' \in \Psi(\mathcal{A})$. There exist $d, d' \in \mathbb{Z}$ such that for any $N \in \mathbb{N}$, $N > |d| + |d'|$, there exist $P, P' \in \mathcal{D}(\mathcal{A})$, $p, p' \in \mathbb{N}_0$, $R \in OP^{-N-d}$, $R' \in OP^{-N-d'}$ such that $T = PD^{-2p} + R$, $T' = P'D^{-2p'} + R'$, $PD^{-2p} \in OP^d$ and $P'D^{-2p'} \in OP^{d'}$.

Thus, $TT' = PD^{-2p}P'D^{-2p'} + RP'D^{-2p'} + PD^{-2p}R' + RR'$.

We also have $RP'D^{-2p'} \in OP^{-N-d'+d} = OP^{-N}$ and similarly, $PD^{-2p}R' \in OP^{-N}$. Since $RR' \in OP^{-2N}$, we get

$$TT' \sim PD^{-2p}P'D^{-2p'} \mod OP^{-N}.$$

If $p = 0$, then $TT' \sim QD^{-2p'} \mod OP^{-N}$ where $Q = PP' \in \mathcal{D}(\mathcal{A})$ and $QD^{-2p'} \in OP^{d+d'}$. Suppose $p \neq 0$. A recurrence proves that for any $q \in \mathbb{N}_0$,

$$D^{-2}P' \sim \sum_{k=0}^q (-1)^k \nabla^k(P')D^{-2k-2} + (-1)^{q+1} D^{-2} \nabla^{q+1}(P')D^{-2q-2} \mod OP^{-\infty}.$$

By Lemma 1.2.5 (v), the remainder is in $OP^{d'+2p'-q-3}$, since $P' \in OP^{d'+2p'}$. Another recurrence gives for any $q \in \mathbb{N}_0$,

$$D^{-2p}P' \sim \sum_{k_1, \dots, k_p=0}^q (-1)^{|k|_1} \nabla^{|k|_1}(P')D^{-2|k|_1-2p} \mod OP^{d'+2p'-q-1-2p}.$$

Thus, with $q_N = N + d + d' - 1$,

$$TT' \sim \sum_{k_1, \dots, k_p=0}^{q_N} (-1)^{|k|_1} P \nabla^{|k|_1} (P') D^{-2|k|_1 - 2(p+p')} \mod OP^{-N}.$$

The last sum can be written $Q_N D^{-2r_N}$ where $r_N := p q_N + (p + p')$. Since $Q_N \in \mathcal{D}(\mathcal{A})$ and $Q_N D^{-2r_N} \in OP^{d+d'}$, the result follows. \square

It is convenient to also introduce

Definition 1.2.9. Let $\mathcal{D}_1(\mathcal{A})$ be the algebra generated by \mathcal{A} , $J\mathcal{A}J^{-1}$ and \mathcal{D} , and $\Psi_1(\mathcal{A})$ be the set of pseudodifferential operators constructed as before with $\mathcal{D}_1(\mathcal{A})$ instead of $\mathcal{D}(\mathcal{A})$. Note that $\Psi_1(\mathcal{A})$ is subalgebra of $\Psi(\mathcal{A})$.

Remark that $\Psi_1(\mathcal{A})$ does not necessarily contain operators such as $|D|^k$ where $k \in \mathbb{Z}$ is odd. This algebra is similar to the one defined in [22].

1.2.2 Zeta functions, noncommutative integral and spectral action

For any operator B and if X is either D or D_A , we define

$$\begin{aligned} \zeta_X^B(s) &:= \text{Tr} (B|X|^{-s}), \\ \zeta_X(s) &:= \text{Tr} (|X|^{-s}). \end{aligned}$$

The dimension spectrum $Sd(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of a spectral triple has been defined in [29, 38]. It is extended here to pay attention to the operator J and to our definition of pseudodifferential operator.

Definition 1.2.10. The spectrum dimension of the spectral triple is the subset $Sd(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of all poles of the functions $\zeta_D^P : s \mapsto \text{Tr} (P|D|^{-s})$ where P is any pseudodifferential operator in OP^0 . The spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is simple when these poles are all simple.

Remark 1.2.11. If $Sp(\mathcal{A}, \mathcal{H}, \mathcal{D})$ denotes the set of all poles of the functions $s \mapsto \text{Tr} (P|D|^{-s})$ where P is any pseudodifferential operator, then, $Sd(\mathcal{A}, \mathcal{H}, \mathcal{D}) \subseteq Sp(\mathcal{A}, \mathcal{H}, \mathcal{D})$.

When $Sp(\mathcal{A}, \mathcal{H}, \mathcal{D}) = \mathbb{Z}$, $Sd(\mathcal{A}, \mathcal{H}, \mathcal{D}) = \{n - k : k \in \mathbb{N}_0\}$: indeed, if P is a pseudodifferential operator in OP^0 , and $q \in \mathbb{N}$ is such that $q > n$, $P|D|^{-s}$ is in $OP^{-\Re(s)}$ so is trace-class for s in a neighborhood of q ; as a consequence, q cannot be a pole of $s \mapsto \text{Tr} (P|D|^{-s})$.

Remark 1.2.12. $Sp(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is also the set of poles of functions $s \mapsto \text{Tr} (B|D|^{-s-2p})$ where $p \in \mathbb{N}_0$ and $B \in \mathcal{D}(\mathcal{A})$.

Introducing the notation (recall that $\nabla(T) = [\mathcal{D}^2, T]$) for an operator T ,

$$\varepsilon(T) := \nabla(T) D^{-2},$$

we get from [22, (2.44)] the following expansion for $T \in OP^q$

$$\sigma_z(T) \sim \sum_{r=0}^N g(z, r) \varepsilon^r(T) \mod OP^{-N-1+q} \quad (1.4)$$

where $g(z, r) := \frac{1}{r!}(\frac{z}{2}) \cdots (\frac{z}{2} - (r-1)) = \binom{z/2}{r}$ with the convention $g(z, 0) := 1$. We define the noncommutative integral by

$$\oint T := \operatorname{Res}_{s=0} \zeta_D^T(s) = \operatorname{Res}_{s=0} \operatorname{Tr} (T|D|^{-s}).$$

The noncommutative integral plays the role of the Wodzicki residue in a spectral triple setting.

Proposition 1.2.13. [38] *If the spectral triple is simple, \oint is a trace on $\Psi(\mathcal{A})$.*

Proof. Let $P \in OP^{k_1}$, $Q \in OP^{k_2} \in \Psi(\mathcal{A})$. With $[Q, |D|^{-s}] = (Q - \sigma_{-s}(Q))|D|^{-s}$ and the equivalence $Q - \sigma_{-s}(Q) \sim -\sum_{r=1}^N g(-s, r) \varepsilon^r(Q) \mod OP^{-N-1+k_2}$, we get

$$P[Q, |D|^{-s}] \sim -\sum_{r=1}^N g(-s, r) P \varepsilon^r(Q) |D|^{-s} \mod OP^{-N-1+k_1+k_2-\Re(s)}$$

which gives, if we choose $N = n + k_1 + k_2$,

$$\operatorname{Res}_{s=0} \operatorname{Tr} (P[Q, |D|^{-s}]) = -\sum_{r=1}^{n+k_1+k_2} \operatorname{Res}_{s=0} g(-s, r) \operatorname{Tr} (P \varepsilon^r(Q) |D|^{-s}).$$

By hypothesis $s \mapsto \operatorname{Tr} (P \varepsilon^r(Q) |D|^{-s})$ has only simple poles. Thus, since $s = 0$ is a zero of the analytic function $s \mapsto g(-s, r)$ for any $r \geq 1$, we have $\operatorname{Res}_{s=0} g(-s, r) \operatorname{Tr} (P \varepsilon^r(Q) |D|^{-s}) = 0$, which entails that $\operatorname{Res}_{s=0} \operatorname{Tr} (P[Q, |D|^{-s}]) = 0$ and thus

$$\oint PQ = \operatorname{Res}_{s=0} \operatorname{Tr} (P|D|^{-s}Q).$$

When $s \in \mathbb{C}$ with $\Re(s) > 2 \max(k_1 + n + 1, k_2)$, the operator $P|D|^{-s/2}$ is trace-class while $|D|^{-s/2}Q$ is bounded, so $\operatorname{Tr} (P|D|^{-s}Q) = \operatorname{Tr} (|D|^{-s/2}QP|D|^{-s/2}) = \operatorname{Tr} (\sigma_{-s/2}(QP)|D|^{-s})$. Thus, using (1.4) again,

$$\operatorname{Res}_{s=0} \operatorname{Tr} (P|D|^{-s}Q) = \oint QP + \sum_{r=1}^{n+k_1+k_2} \operatorname{Res}_{s=0} g(-s/2, r) \operatorname{Tr} (\varepsilon^r(QP)|D|^{-s}).$$

As before, for any $r \geq 1$, $\operatorname{Res}_{s=0} g(-s/2, r) \operatorname{Tr} (\varepsilon^r(QP)|D|^{-s}) = 0$ since $g(0, r) = 0$ and the spectral triple is simple. Finally,

$$\operatorname{Res}_{s=0} \operatorname{Tr} (P|D|^{-s}Q) = \oint QP.$$

which yields the result. \square

On a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, the role of the action is played by the “spectral action” as introduced by A. Chamseddine and A. Connes:

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) := \operatorname{Tr} (\Phi(\mathcal{D}_A/\Lambda)) \quad (1.5)$$

where Φ is any even positive cut-off function which could be replaced by a step function up to some mathematical difficulties investigated in [54] and Λ fixes the mass scale. This means that \mathcal{S} counts the spectral values of $|\mathcal{D}_A|$ less than the mass scale Λ (note that the resolvent set of \mathcal{D}_A is compact since, by assumption, the same is true for \mathcal{D} , see Lemma 1.2.3 below). The Chamseddine–Connes spectral action principle asserts that (see [37, p. 197]) the spectral action *is the fundamental action functional S that can be used both at the classical level to compare different geometric spaces and at the quantum level in the functional integral formulation, after Wick rotation to Euclidean signature*. In other words, the functional $\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda)$ which is related to the spectrum of the Dirac operator \mathcal{D} , contains all physical information of the (geometrized) quantum field theory associated to the triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. It is therefore crucial to be able to compute it on some fundamental examples. This spectral action is known on few examples: [17, 22, 24, 37, 53, 60–62, 69, 93]. We will study the case of the noncommutative torus in chapter 2, and the case of $SU_q(2)$ in chapter 3. We shall investigate in chapter 4 some questions about tadpoles, which are the linear terms in A in the spectral action. In the case of a spectral triple with simple dimension spectrum, we have (see for instance [37, Theorem 1.145])

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = \sum_{0 < k \in Sd^+} \Phi_k \Lambda^k \oint |D_A|^{-k} + \Phi(0) \zeta_{D_A}(0) + \mathcal{O}(\Lambda^{-1}), \quad (1.6)$$

where $\Phi_k = \frac{1}{2} \int_0^\infty \Phi(t) t^{k/2-1} dt$ and Sd^+ is the strictly positive part of the spectrum dimension of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. Thus, the main problem is the computation of the $\oint |D_A|^{-k}$, $\zeta_{D_A}(0)$ terms. We consider this question in the following section.

1.3 Residues of ζ_{D_A} for a spectral triple with simple dimension spectrum

We fix a regular real spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, J)$ of dimension n and a self-adjoint 1-form A . Recall that

$$\begin{aligned} \mathcal{D}_A &:= \mathcal{D} + \tilde{A} \text{ where } \tilde{A} := A + \varepsilon J A J^{-1}, \\ D_A &:= \mathcal{D}_A + P_A \end{aligned}$$

where P_A is the projection on $\text{Ker } \mathcal{D}_A$. Remark that $\tilde{A} \in \mathcal{D}(\mathcal{A}) \cap OP^0$ and $\mathcal{D}_A \in \mathcal{D}(\mathcal{A}) \cap OP^1$.

We denote

$$V_A := P_A - P_0.$$

As the following lemma shows, V_A is a smoothing operator:

Lemma 1.3.1. (i) $\bigcap_{k \geq 1} \text{Dom}(\mathcal{D}_A)^k \subseteq \bigcap_{k \geq 1} \text{Dom } |D|^k$.

(ii) $\text{Ker } \mathcal{D}_A \subseteq \bigcap_{k \geq 1} \text{Dom } |D|^k$.

(iii) For any $\alpha, \beta \in \mathbb{R}$, $|D|^\beta P_A |D|^\alpha$ is bounded.

(iv) $P_A \in OP^{-\infty}$.

Proof. (i) Let us define for any $p \in \mathbb{N}$, $R_p := (\mathcal{D}_A)^p - \mathcal{D}^p$, so $R_p \in OP^{p-1}$ and $R_p(\text{Dom } |D|^p) \subseteq \text{Dom } |D|$ (see Remark 1.2.6).

Let us fix $k \in \mathbb{N}$, $k \geq 2$. Since $\text{Dom } \mathcal{D}_A = \text{Dom } \mathcal{D} = \text{Dom } |D|$, we have

$$\text{Dom}(\mathcal{D}_A)^k = \{ \phi \in \text{Dom } |D| : (\mathcal{D}^j + R_j) \phi \in \text{Dom } |D|, \forall j \text{ } 1 \leq j \leq k-1 \}.$$

Let $\phi \in \text{Dom}(\mathcal{D}_A)^k$. We prove by recurrence that for any $j \in \{1, \dots, k-1\}$, $\phi \in \text{Dom}|D|^{j+1}$:

We have $\phi \in \text{Dom}|D|$ and $(\mathcal{D} + R_1)\phi \in \text{Dom}|D|$. Thus, since $R_1\phi \in \text{Dom}|D|$, $\mathcal{D}\phi \in \text{Dom}|D|$, which proves that $\phi \in \text{Dom}|D|^2$. Hence, case $j = 1$ is done.

Suppose now that $\phi \in \text{Dom}|D|^{j+1}$ for a $j \in \{1, \dots, k-2\}$. Since $(\mathcal{D}^{j+1} + R_{j+1})\phi \in \text{Dom}|D|$, and $R_{j+1}\phi \in \text{Dom}|D|$, we get $\mathcal{D}^{j+1}\phi \in \text{Dom}|D|$, which proves that $\phi \in \text{Dom}|D|^{j+2}$.

Finally, if we set $j = k-1$, we get $\phi \in \text{Dom}|D|^k$, so $\text{Dom}(\mathcal{D}_A)^k \subseteq \text{Dom}|D|^k$.

(ii) follows from $\text{Ker } \mathcal{D}_A \subseteq \bigcap_{k \geq 1} \text{Dom}(\mathcal{D}_A)^k$ and (i).

(iii) Let us first check that $|D|^\alpha P_A$ is bounded. We define D_0 as the operator with domain $\text{Dom } D_0 = \text{Im } P_A \cap \text{Dom}|D|^\alpha$ and such that $D_0\phi = |D|^\alpha\phi$. Since $\text{Dom } D_0$ is finite dimensional, D_0 extends as a bounded operator on \mathcal{H} with finite rank. We have

$$\sup_{\phi \in \text{Dom}|D|^\alpha P_A, \|\phi\| \leq 1} \||D|^\alpha P_A \phi\| \leq \sup_{\phi \in \text{Dom } D_0, \|\phi\| \leq 1} \||D|^\alpha \phi\| = \|D_0\| < \infty$$

so $|D|^\alpha P_A$ is bounded. We can remark that by (ii), $\text{Dom } D_0 = \text{Im } P_A$ and $\text{Dom}|D|^\alpha P_A = \mathcal{H}$.

Let us prove now that $P_A|D|^\alpha$ is bounded: Let $\phi \in \text{Dom } P_A|D|^\alpha = \text{Dom}|D|^\alpha$. By (ii), we have $\text{Im } P_A \subseteq \text{Dom}|D|^\alpha$ so we get

$$\begin{aligned} \|P_A|D|^\alpha \phi\| &\leq \sup_{\psi \in \text{Im } P_A, \|\psi\| \leq 1} |\langle \psi, |D|^\alpha \phi \rangle| \leq \sup_{\psi \in \text{Im } P_A, \|\psi\| \leq 1} |\langle |D|^\alpha \psi, \phi \rangle| \\ &\leq \sup_{\psi \in \text{Im } P_A, \|\psi\| \leq 1} \||D|^\alpha \psi\| \|\phi\| = \|D_0\| \|\phi\|. \end{aligned}$$

(iv) For any $k \in \mathbb{N}_0$ and $t \in \mathbb{R}$, $\delta^k(P_A)|D|^t$ is a linear combination of terms of the form $|D|^\beta P_A|D|^\alpha$, so the result follows from (iii). \square

Remark 1.3.2. We will see later on the noncommutative torus example how important is the difference between \mathcal{D}_A and $\mathcal{D} + A$. In particular, the inclusion $\text{Ker } \mathcal{D} \subseteq \text{Ker } \mathcal{D} + A$ is not satisfied since A does not preserve $\text{Ker } \mathcal{D}$ contrarily to \tilde{A} .

The coefficient of the nonconstant term Λ^k ($k > 0$) in the expansion (1.6) of the spectral action $S(\mathcal{D}_A, \Phi, \Lambda)$ is equal to the residue of $\zeta_{D_A}(s)$ at k . We will see in this section how we can compute these residues in term of noncommutative integral of certain operators.

Define for any operator T , $p \in \mathbb{N}$, $s \in \mathbb{C}$,

$$K_p(T, s) := \left(-\frac{s}{2}\right)^p \int_{0 \leq t_1 \leq \dots \leq t_p \leq 1} \sigma_{-st_1}(T) \cdots \sigma_{-st_p}(T) dt$$

with $dt := dt_1 \cdots dt_p$.

Remark that if $T \in OP^\alpha$, then $\sigma_z(T) \in OP^\alpha$ for $z \in \mathbb{C}$ and $K_p(T, s) \in OP^{\alpha p}$.

Let us define

$$\begin{aligned} X &:= \mathcal{D}_A^2 - \mathcal{D}^2 = \tilde{A}\mathcal{D} + \mathcal{D}\tilde{A} + \tilde{A}^2, \\ X_V &:= X + V_A, \end{aligned}$$

thus $X \in \mathcal{D}_1(\mathcal{A}) \cap OP^1$ and by Lemma 1.3.1,

$$X_V \sim X \mod OP^{-\infty}. \quad (1.7)$$

We will use

$$Y := \log(D_A^2) - \log(D^2)$$

which makes sense since $D_A^2 = \mathcal{D}_A^2 + P_A$ is invertible for any A .

By definition of X_V , we get

$$Y = \log(D^2 + X_V) - \log(D^2).$$

Remark that most of the results that we will present here are based on the fact that $D_A - D$ is a pseudodifferential operator of order 0.

Lemma 1.3.3. [22]

(i) Y is a pseudodifferential operator in OP^{-1} with the following expansion for any $N \in \mathbb{N}$

$$Y \sim \sum_{p=1}^N \sum_{k_1, \dots, k_p=0}^{N-p} \frac{(-1)^{|k|_1+p+1}}{|k|_1+p} \nabla^{k_p} (X \nabla^{k_{p-1}} (\dots X \nabla^{k_1} (X) \dots)) D^{-2(|k|_1+p)} \mod OP^{-N-1}.$$

(ii) For any $N \in \mathbb{N}$ and $s \in \mathbb{C}$,

$$|D_A|^{-s} \sim |D|^{-s} + \sum_{p=1}^N K_p(Y, s) |D|^{-s} \mod OP^{-N-1-\Re(s)}. \quad (1.8)$$

Proof. (i) We follow [22, Lemma 2.2]. By functional calculus, $Y = \int_0^\infty I(\lambda) d\lambda$, where

$$I(\lambda) \sim \sum_{p=1}^N (-1)^{p+1} ((D^2 + \lambda)^{-1} X_V)^p (D^2 + \lambda)^{-1} \mod OP^{-N-3}.$$

By (1.7), $((D^2 + \lambda)^{-1} X_V)^p \sim ((D^2 + \lambda)^{-1} X)^p \mod OP^{-\infty}$ and we get

$$I(\lambda) \sim \sum_{p=1}^N (-1)^{p+1} ((D^2 + \lambda)^{-1} X)^p (D^2 + \lambda)^{-1} \mod OP^{-N-3}.$$

We set $A_p(X) := ((D^2 + \lambda)^{-1} X)^p (D^2 + \lambda)^{-1}$ and $L := (D^2 + \lambda)^{-1} \in OP^{-2}$ for a fixed λ . Since $[D^2 + \lambda, X] \sim \nabla(X) \mod OP^{-\infty}$, a recurrence proves that if T is an operator in OP^r , then, for $q \in \mathbb{N}_0$,

$$A_1(T) = LTL \sim \sum_{k=0}^q (-1)^k \nabla^k(T) L^{k+2} \mod OP^{r-q-5}.$$

With $A_p(X) = LX A_{p-1}(X)$, another recurrence gives, for any $q \in \mathbb{N}_0$,

$$A_p(X) \sim \sum_{k_1, \dots, k_p=0}^q (-1)^{|k|_1} \nabla^{k_p} (X \nabla^{k_{p-1}} (\dots X \nabla^{k_1} (X) \dots)) L^{|k|_1+p+1} \mod OP^{-q-p-3},$$

which entails that

$$I(\lambda) \sim \sum_{p=1}^N (-1)^{p+1} \sum_{k_1, \dots, k_p=0}^{N-p} (-1)^{|k|_1} \nabla^{k_p} (X \nabla^{k_{p-1}} (\dots X \nabla^{k_1} (X) \dots)) L^{|k|_1+p+1} \mod OP^{-N-3}.$$

With $\int_0^\infty (D^2 + \lambda)^{-(|k|_1+p+1)} d\lambda = \frac{1}{|k|_1+p} D^{-2(|k|_1+p)}$, we get the result provided we control the remainders. Such a control is given in [22, (2.27)].

(ii) We have $|D_A|^{-s} = e^{B-(s/2)Y} e^{-B} |D|^{-s}$ where $B := (-s/2) \log(D^2)$. Following [22, Theorem 2.4], we get

$$|D_A|^{-s} = |D|^{-s} + \sum_{p=1}^{\infty} K_p(Y, s) |D|^{-s}. \quad (1.9)$$

and each $K_p(Y, s)$ is in OP^{-p} . \square

Corollary 1.3.4. *For any $p \in \mathbb{N}$ and $r_1, \dots, r_p \in \mathbb{N}_0$, $\varepsilon^{r_1}(Y) \cdots \varepsilon^{r_p}(Y) \in \Psi_1(\mathcal{A})$.*

Proof. If for any $q \in \mathbb{N}$ and $k = (k_1, \dots, k_q) \in \mathbb{N}_0^q$,

$$\Gamma_q^k(X) := \frac{(-1)^{|k|_1+q+1}}{|k|_1+q} \nabla^{k_q} (X \nabla^{k_{q-1}} (\dots X \nabla^{k_1}(X) \dots)),$$

then, $\Gamma_q^k(X) \in OP^{|k|_1+q}$. For any $N \in \mathbb{N}$,

$$Y \sim \sum_{q=1}^N \sum_{k_1, \dots, k_q=0}^{N-q} \Gamma_q^k(X) D^{-2(|k|_1+q)} \mod OP^{-N-1}. \quad (1.10)$$

Note that the $\Gamma_q^k(X)$ are in $\mathcal{D}_1(\mathcal{A})$, which, with (1.10) proves that Y and thus $\varepsilon^r(Y) = \nabla^r(Y) D^{-2r}$, are also in $\Psi_1(\mathcal{A})$. \square

We remark, as in [32], that the fluctuations leave invariant the first term of the spectral action (1.6). This is a generalization of the fact that in the commutative case, the noncommutative integral $\oint |D|^{-n}$, where n is the dimension of the manifold, only depends on the principal symbol of the Dirac operator \mathcal{D} and this symbol is stable by adding a gauge potential like in $\mathcal{D} + A$. Note however that the symmetrized gauge potential $A + \epsilon J A J^{-1}$ is always zero in this case for any selfadjoint one-form A .

Lemma 1.3.5. *If the spectral triple is simple,*

$$\zeta_{D_A}(0) - \zeta_D(0) = \sum_{q=1}^n \frac{(-1)^q}{q} \oint (\tilde{A} D^{-1})^q. \quad (1.11)$$

Proof. Since the spectral triple is simple, equation (1.9) entails that

$$\zeta_{D_A}(0) - \zeta_D(0) = \text{Tr}(K_1(Y, s) |D|^{-s})_{s=0}.$$

Thus, with (1.4), we get $\zeta_{D_A}(0) - \zeta_D(0) = -\frac{1}{2} \oint Y$. Replacing A by \tilde{A} , the same proof as in [22] gives

$$-\frac{1}{2} \oint Y = \sum_{q=1}^n \frac{(-1)^q}{q} \oint (\tilde{A} D^{-1})^q. \quad \square$$

Lemma 1.3.6. *For any $k \in \mathbb{N}_0$,*

$$\operatorname{Res}_{s=n-k} \zeta_{D_A}(s) = \operatorname{Res}_{s=n-k} \zeta_D(s) + \sum_{p=1}^k \sum_{r_1, \dots, r_p=0}^{k-p} \operatorname{Res}_{s=n-k} h(s, r, p) \operatorname{Tr}(\varepsilon^{r_1}(Y) \cdots \varepsilon^{r_p}(Y) |D|^{-s}),$$

where

$$h(s, r, p) := (-s/2)^p \int_{0 \leq t_1 \leq \dots \leq t_p \leq 1} g(-st_1, r_1) \cdots g(-st_p, r_p) dt.$$

Proof. By Lemma 1.3.3 (ii), $|D_A|^{-s} \sim |D|^{-s} + \sum_{p=1}^k K_p(Y, s) |D|^{-s} \pmod{OP^{-(k+1)-\Re(s)}}$, where the convention $\sum_{\emptyset} = 0$ is used. Thus, we get for s in a neighborhood of $n - k$,

$$|D_A|^{-s} - |D|^{-s} - \sum_{p=1}^k K_p(Y, s) |D|^{-s} \in OP^{-(k+1)-\Re(s)} \subseteq \mathcal{L}^1(\mathcal{H})$$

which gives

$$\operatorname{Res}_{s=n-k} \zeta_{D_A}(s) = \operatorname{Res}_{s=n-k} \zeta_D(s) + \sum_{p=1}^k \operatorname{Res}_{s=n-k} \operatorname{Tr}(K_p(Y, s) |D|^{-s}). \quad (1.12)$$

Let us fix $1 \leq p \leq k$ and $N \in \mathbb{N}$. By (1.4) we get

$$K_p(Y, s) \sim (-\tfrac{s}{2})^p \int_{0 \leq t_1 \leq \dots \leq t_p \leq 1} \sum_{r_1, \dots, r_p=0}^N g(-st_1, r_1) \cdots g(-st_p, r_p) \varepsilon^{r_1}(Y) \cdots \varepsilon^{r_p}(Y) dt \pmod{OP^{-N-p-1}}. \quad (1.13)$$

If we now take $N = k - p$, we get for s in a neighborhood of $n - k$

$$K_p(Y, s) |D|^{-s} - \sum_{r_1, \dots, r_p=0}^{k-p} h(s, r, p) \varepsilon^{r_1}(Y) \cdots \varepsilon^{r_p}(Y) |D|^{-s} \in OP^{-k-1-\Re(s)} \subseteq \mathcal{L}^1(\mathcal{H})$$

so (1.12) gives the result. \square

Our operators $|D_A|^k$ are pseudodifferential operators:

Lemma 1.3.7. *For any $k \in \mathbb{Z}$, $|D_A|^k \in \Psi^k(\mathcal{A})$.*

Proof. Using (1.13), we see that $K_p(Y, s)$ is a pseudodifferential operator in OP^{-p} , so (1.8) proves that $|D_A|^k$ is a pseudodifferential operator in OP^k . \square

The following result is quite important since it shows that one can use \oint for D or D_A :

Proposition 1.3.8. *If the spectral triple is simple, $\operatorname{Res}_{s=0} \operatorname{Tr}(P |D_A|^{-s}) = \oint P$ for any pseudodifferential operator P . In particular, for any $k \in \mathbb{N}_0$*

$$\oint |D_A|^{-(n-k)} = \operatorname{Res}_{s=n-k} \zeta_{D_A}(s).$$

Proof. Suppose $P \in OP^k$ with $k \in \mathbb{Z}$ and let us fix $p \geq 1$. With (1.13), we see that for any $N \in \mathbb{N}$,

$$PK_p(Y, s)|D|^{-s} \sim \sum_{r_1, \dots, r_p=0}^N h(s, r, p) P\varepsilon^{r_1}(Y) \cdots \varepsilon^{r_p}(Y)|D|^{-s} \mod OP^{-N-p-1+k-\Re(s)}.$$

Thus if we take $N = n - p + k$, we get

$$\operatorname{Res}_{s=0} \operatorname{Tr} (PK_p(Y, s)|D|^{-s}) = \sum_{r_1, \dots, r_p=0}^{n-p+k} \operatorname{Res}_{s=0} h(s, r, p) \operatorname{Tr} (P\varepsilon^{r_1}(Y) \cdots \varepsilon^{r_p}(Y)|D|^{-s}).$$

Since $s = 0$ is a zero of the analytic function $s \mapsto h(s, r, p)$ and $s \mapsto \operatorname{Tr} P\varepsilon^{r_1}(Y) \cdots \varepsilon^{r_p}(Y)|D|^{-s}$ has only simple poles by hypothesis, we see that $\operatorname{Res}_{s=0} h(s, r, p) \operatorname{Tr} (P\varepsilon^{r_1}(Y) \cdots \varepsilon^{r_p}(Y)|D|^{-s}) = 0$ and

$$\operatorname{Res}_{s=0} \operatorname{Tr} (PK_p(Y, s)|D|^{-s}) = 0. \quad (1.14)$$

Using (1.8), $P|D_A|^{-s} \sim P|D|^{-s} + \sum_{p=1}^{k+n} PK_p(Y, s)|D|^{-s} \mod OP^{-n-1-\Re(s)}$ and thus,

$$\operatorname{Res}_{s=0} \operatorname{Tr} (P|D_A|^{-s}) = \oint P + \sum_{p=1}^{k+n} \operatorname{Res}_{s=0} \operatorname{Tr} (PK_p(Y, s)|D|^{-s}). \quad (1.15)$$

The result now follows from (1.14) and (1.15). To get the last equality, one uses the pseudodifferential operator $|D_A|^{-(n-k)}$. \square

Proposition 1.3.9. *If the spectral triple is simple, then*

$$\oint |D_A|^{-n} = \oint |D|^{-n}. \quad (1.16)$$

Proof. Lemma 1.3.6 and previous proposition for $k = 0$. \square

Lemma 1.3.10. *If the spectral triple is simple,*

- (i) $\oint |D_A|^{-(n-1)} = \oint |D|^{-(n-1)} - \left(\frac{n-1}{2}\right) \oint X|D|^{-n-1}.$
- (ii) $\oint |D_A|^{-(n-2)} = \oint |D|^{-(n-2)} + \frac{n-2}{2} \left(- \oint X|D|^{-n} + \frac{n}{4} \oint X^2|D|^{-2-n} \right).$

Proof. (i) By (1.8),

$$\operatorname{Res}_{s=n-1} \zeta_{D_A}(s) - \zeta_D(s) = \operatorname{Res}_{s=n-1} (-s/2) \operatorname{Tr} (Y|D|^{-s}) = -\frac{n-1}{2} \operatorname{Res}_{s=0} \operatorname{Tr} (Y|D|^{-(n-1)}|D|^{-s})$$

where for the last equality we use the simple dimension spectrum hypothesis. Lemma 1.3.3 (i) yields $Y \sim XD^{-2} \mod OP^{-2}$ and $Y|D|^{-(n-1)} \sim X|D|^{-n-1} \mod OP^{-n-1} \subseteq \mathcal{L}^1(\mathcal{H})$. Thus,

$$\operatorname{Res}_{s=0} \operatorname{Tr} (Y|D|^{-(n-1)}|D|^{-s}) = \operatorname{Res}_{s=0} \operatorname{Tr} (X|D|^{-n-1}|D|^{-s}) = \oint X|D|^{-n-1}.$$

(ii) Lemma 1.3.6 (ii) gives

$$\operatorname{Res}_{s=n-2} \zeta_{D_A}(s) = \operatorname{Res}_{s=n-2} \zeta_D(s) + \operatorname{Res}_{s=n-2} \sum_{r=0}^1 h(s, r, 1) \operatorname{Tr}(\varepsilon^r(Y)|D|^{-s}) + h(s, 0, 2) \operatorname{Tr}(Y^2|D|^{-s}).$$

We have $h(s, 0, 1) = -\frac{s}{2}$, $h(s, 1, 1) = \frac{1}{2}(\frac{s}{2})^2$ and $h(s, 0, 2) = \frac{1}{2}(\frac{s}{2})^2$. Using again Lemma 1.3.3 (i),

$$Y \sim XD^{-2} - \frac{1}{2}\nabla(X)D^{-4} - \frac{1}{2}X^2D^{-4} \mod OP^{-3}.$$

Thus,

$$\operatorname{Res}_{s=n-2} \operatorname{Tr}(Y|D|^{-s}) = \oint X|D|^{-n} - \frac{1}{2} \oint (\nabla(X) + X^2)|D|^{-2-n}.$$

Moreover, using $\oint \nabla(X)|D|^{-k} = 0$ for any $k \geq 0$ since \oint is a trace,

$$\operatorname{Res}_{s=n-2} \operatorname{Tr}(\varepsilon(Y)|D|^{-s}) = \operatorname{Res}_{s=n-2} \operatorname{Tr}(\nabla(X)D^{-4}|D|^{-s}) = \oint \nabla(X)|D|^{-2-n} = 0.$$

Similarly, since $Y \sim XD^{-2} \mod OP^{-2}$ and $Y^2 \sim X^2D^{-4} \mod OP^{-3}$, we get

$$\operatorname{Res}_{s=n-2} \operatorname{Tr}(Y^2|D|^{-s}) = \operatorname{Res}_{s=n-2} \operatorname{Tr}(X^2D^{-4}|D|^{-s}) = \oint X^2|D|^{-2-n}.$$

Thus,

$$\begin{aligned} \operatorname{Res}_{s=n-2} \zeta_{D_A}(s) &= \operatorname{Res}_{s=n-2} \zeta_D(s) + (-\frac{n-2}{2}) \left(\oint X|D|^{-n} - \frac{1}{2} \oint (\nabla(X) + X^2)|D|^{-2-n} \right) \\ &\quad + \frac{1}{2}(\frac{n-2}{2})^2 \oint \nabla(X)|D|^{-2-n} + \frac{1}{2}(\frac{n-2}{2})^2 \oint X^2|D|^{-2-n}. \end{aligned}$$

Finally,

$$\operatorname{Res}_{s=n-2} \zeta_{D_A}(s) = \operatorname{Res}_{s=n-2} \zeta_D(s) + (-\frac{n-2}{2}) \left(\oint X|D|^{-n} - \frac{1}{2} \oint X^2|D|^{-2-n} \right) + \frac{1}{2}(\frac{n-2}{2})^2 \oint X^2|D|^{-2-n}$$

and the result follows from Proposition 1.3.8. \square

Corollary 1.3.11. *If the spectral triple is simple and satisfies $\oint |D|^{-(n-2)} = \oint \tilde{A}\mathcal{D}|D|^{-n} = \oint \mathcal{D}\tilde{A}|D|^{-n} = 0$, then*

$$\oint |D_A|^{-(n-2)} = \frac{n(n-2)}{4} \left(\oint \tilde{A}\mathcal{D}\tilde{A}\mathcal{D}|D|^{-n-2} + \frac{n-2}{n} \oint \tilde{A}^2|D|^{-n} \right).$$

Proof. By previous lemma,

$$\operatorname{Res}_{s=n-2} \zeta_{D_A}(s) = \frac{n-2}{2} \left(- \oint \tilde{A}^2|D|^{-n} + \frac{n}{4} \oint (\tilde{A}\mathcal{D}\tilde{A}\mathcal{D} + \mathcal{D}\tilde{A}\mathcal{D}\tilde{A} + \tilde{A}\mathcal{D}^2\tilde{A} + \mathcal{D}\tilde{A}^2\mathcal{D})|D|^{-n-2} \right).$$

Since $\nabla(\tilde{A}) \in OP^1$, the trace property of \oint yields the result. \square

1.4 Noncommutative integrals and tadpoles

Let $(\mathcal{A}, \mathcal{D}, \mathcal{H}, J)$ be a real regular spectral triple of dimension d . Recall that a one-form A is a finite sum of operators like $a_1[\mathcal{D}, a_2]$ where $a_i \in \mathcal{A}$. The set of one-forms is denoted by $\Omega_{\mathcal{D}}^1(\mathcal{A})$.

Lemma 1.4.1. *Let $(\mathcal{A}, \mathcal{D}, \mathcal{H})$ be a spectral triple and $X \in \Psi(\mathcal{A})$. Then*

$$\oint X^* = \overline{\oint X}.$$

If the spectral triple is real, then, for $X \in \Psi(\mathcal{A})$, $JXJ^{-1} \in \Psi(\mathcal{A})$ and

$$\oint JXJ^{-1} = \oint X^* = \overline{\oint X}.$$

Proof. The first result follows from (for s large enough, so the operators are traceable)

$$\mathrm{Tr}(X^*|\mathcal{D}|^{-s}) = \mathrm{Tr}((|\mathcal{D}|^{-\bar{s}}X)^*) = \overline{\mathrm{Tr}(|\mathcal{D}|^{-\bar{s}}X)} = \overline{\mathrm{Tr}(X|\mathcal{D}|^{-\bar{s}})}.$$

The second result is due to the anti-linearity of J , $\mathrm{Tr}(JYJ^{-1}) = \overline{\mathrm{Tr}(Y)}$, and $J|\mathcal{D}| = |\mathcal{D}|J$, so

$$\mathrm{Tr}(X|\mathcal{D}|^{-s}) = \overline{\mathrm{Tr}(JX|\mathcal{D}|^{-s}J^{-1})} = \overline{\mathrm{Tr}(JXJ^{-1}|\mathcal{D}|^{-\bar{s}})}.$$

□

Corollary 1.4.2. *For any one-form $A = A^*$, and for $k, l \in \mathbb{N}$,*

$$\oint A^l \mathcal{D}^{-k} \in \mathbb{R}, \quad \oint (A\mathcal{D}^{-1})^k \in \mathbb{R}, \quad \oint A^l |\mathcal{D}|^{-k} \in \mathbb{R}, \quad \oint \chi A^l |\mathcal{D}|^{-k} \in \mathbb{R}, \quad \oint A^l \mathcal{D} |\mathcal{D}|^{-k} \in \mathbb{R}.$$

In [37], is introduced the following

Definition 1.4.3. In $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, the tadpole $\mathrm{Tad}_{\mathcal{D}+A}(k)$ of order k , for $k \in \{d-l : l \in \mathbb{N}\}$ is the term linear in $A = A^* \in \Omega_{\mathcal{D}}^1$, in the Λ^k term of (1.6) (considered as an infinite series) where $\mathcal{D}_A = \mathcal{D} + A$.

If moreover, the triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, J)$ is real, the tadpole $\mathrm{Tad}_{\mathcal{D}+\tilde{A}}(k)$ is the term linear in A , in the Λ^k term of (1.6) where $\mathcal{D}_A = \mathcal{D} + \tilde{A}$.

Proposition 1.4.4. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple of dimension d with simple dimension spectrum. Then*

$$\mathrm{Tad}_{\mathcal{D}+A}(d-k) = -(d-k) \oint A\mathcal{D}|\mathcal{D}|^{-(d-k)-2}, \quad \forall k \neq d, \quad (1.17)$$

$$\mathrm{Tad}_{\mathcal{D}+A}(0) = -\oint A\mathcal{D}^{-1}. \quad (1.18)$$

Moreover, if the triple is real, $\mathrm{Tad}_{\mathcal{D}+\tilde{A}} = 2 \mathrm{Tad}_{\mathcal{D}+A}$.

Proof. By Lemma 1.3.6 and Proposition 1.3.8, we have the following formula, for any $k \in \mathbb{N}$,

$$\oint |\mathcal{D}_A|^{-(d-k)} = \oint |\mathcal{D}|^{-(d-k)} + \sum_{p=1}^k \sum_{r_1, \dots, r_p=0}^{k-p} \mathrm{Res}_{s=d-k} h(s, r, p) \mathrm{Tr}(\varepsilon^{r_1}(Y) \cdots \varepsilon^{r_p}(Y) |\mathcal{D}|^{-s}),$$

where

$$\begin{aligned}
h(s, r, p) &:= (-s/2)^p \int_{0 \leq t_1 \leq \dots \leq t_p \leq 1} g(-st_1, r_1) \cdots g(-st_p, r_p) dt, \\
\varepsilon^r(T) &:= \nabla(T) \mathcal{D}^{-2r}, \quad \nabla(T) := [\mathcal{D}^2, T], \\
g(z, r) &:= \binom{z/2}{r} \text{ with } g(z, 0) := 1, \\
Y &\sim \sum_{q=1}^N \sum_{k_1, \dots, k_q=0}^{N-q} \Gamma_q^k(X) \mathcal{D}^{-2(|k|_1+q)} \mod OP^{-N-1} \text{ for any } N \in \mathbb{N}^*, \\
X &:= \tilde{A} \mathcal{D} + \mathcal{D} \tilde{A} + \tilde{A}^2, \quad \tilde{A} := A + \epsilon J A J^{-1}, \\
\Gamma_q^k(X) &:= \frac{(-1)^{|k|_1+q+1}}{|k|_1+q} \nabla^{k_q} (X \nabla^{k_{q-1}} (\dots X \nabla^{k_1} (X) \dots)), \quad \forall q \in \mathbb{N}^*, \quad k = (k_1, \dots, k_q) \in \mathbb{N}^q.
\end{aligned}$$

As a consequence, for $k \neq n$, only the terms with $p = 1$ contribute to the linear part:

$$\text{Tad}_{\mathcal{D}+\tilde{A}}(d-k) = \text{Lin}_A \left(\oint |\mathcal{D}_A|^{-(d-k)} \right) = \sum_{r=0}^{k-1} \text{Res}_{s=d-k} h(s, r, 1) \text{Tr} (\varepsilon^r(\text{Lin}_A(Y)) |\mathcal{D}|^{-s}).$$

We check that for any $N \in \mathbb{N}^*$,

$$\text{Lin}_A(Y) \sim \sum_{l=0}^{N-1} \Gamma_1^l(\tilde{A} \mathcal{D} + \mathcal{D} \tilde{A}) \mathcal{D}^{-2(l+1)} \mod OP^{-N-1}.$$

Since $\Gamma_1^l(\tilde{A} \mathcal{D} + \mathcal{D} \tilde{A}) = \frac{(-1)^l}{l+1} \nabla^l(\tilde{A} \mathcal{D} + \mathcal{D} \tilde{A}) = \frac{(-1)^l}{l+1} \{\nabla^l(\tilde{A}), \mathcal{D}\}$, we get, assuming the dimension spectrum to be simple

$$\begin{aligned}
\text{Tad}_{\mathcal{D}+\tilde{A}}(d-k) &= \sum_{r=0}^{k-1} \text{Res}_{s=d-k} h(s, r, p) \text{Tr} (\varepsilon^r(\text{Lin}_A(Y)) |\mathcal{D}|^{-s}) \\
&= \sum_{r=0}^{k-1} h(d-k, r, 1) \sum_{l=0}^{k-1-r} \frac{(-1)^l}{l+1} \text{Res}_{s=d-k} \text{Tr} (\varepsilon^r(\{\nabla^l(\tilde{A}), \mathcal{D}\}) |\mathcal{D}|^{-s-2(l+1)}) \\
&= 2 \sum_{r=0}^{k-1} h(d-k, r, 1) \sum_{l=0}^{k-1-r} \frac{(-1)^l}{l+1} \oint \nabla^{r+l}(\tilde{A}) \mathcal{D} |\mathcal{D}|^{-(d-k+2(r+l))-2} \\
&= -(d-k) \oint \tilde{A} \mathcal{D} |\mathcal{D}|^{-(d-k)-2},
\end{aligned}$$

because in the last sum, only the case $r+l=0$ remains, so $r=l=0$.

Formula (1.18) is a direct application of (1.11).

The link between $\text{Tad}_{\mathcal{D}+\tilde{A}}$ and $\text{Tad}_{\mathcal{D}+A}$ follows from $J\mathcal{D} = \epsilon \mathcal{D} J$ and Lemma 1.4.1. \square

Corollary 1.4.5. *In a real spectral triple $(\mathcal{A}, H, \mathcal{D})$, if $A = A^* \in \Omega_{\mathcal{D}}^1(\mathcal{A})$ is such that $\tilde{A} = 0$, then $\text{Tad}_{\mathcal{D}+A}(k) = 0$ for any $k \in \mathbb{Z}$, $k \leq d$.*

Remark 1.4.6. *Note that $\tilde{A} = 0$ for all $A = A^* \in \Omega_{\mathcal{D}}^1$, when \mathcal{A} is commutative and $JaJ^{-1} = a^*$, for all $a \in \mathcal{A}$, see (4.10), so one can only use $\mathcal{D}_A = \mathcal{D} + A$.*

But we can have \mathcal{A} commutative and $JaJ^{-1} \neq a^*$ [33, 94]:
 Let $\mathcal{A}_1 = \mathbb{C} \oplus \mathbb{C}$ represented on $\mathcal{H}_1 = \mathbb{C}^3$ with, for some complex number $m \neq 0$,

$$\pi_1(a) := \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_1 & 0 \\ 0 & 0 & b_2 \end{pmatrix}, \text{ for } a = (b_1, b_2) \in \mathcal{A}$$

$$\mathcal{D}_1 := \begin{pmatrix} 0 & m & m \\ \bar{m} & 0 & 0 \\ \bar{m} & 0 & b \end{pmatrix}, \chi_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, J_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \circ cc$$

where cc is the complex conjugation. Then $(\mathcal{A}_1, \mathcal{H}_1, \mathcal{D}_1)$ is a commutative real spectral triple of dimension $d = 0$ with non zero one-forms and such that $J_1 \pi_1(a) J_1^{-1} = \pi_1(a^*)$ only if $a = (b_1, b_1)$.

Take a commutative geometry $(\mathcal{A}_2 = C^\infty(M), \mathcal{H} = L^2(M, S), \mathcal{D}_2, \chi_2, J_2)$ defined in 4.3.1 where $d = \dim M$ is even, and then the tensor product of the two spectral triples, namely $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, $\mathcal{D} = \mathcal{D}_1 \otimes \chi_2 + 1 \otimes \mathcal{D}_2$, $\chi = \chi_1 \otimes \chi_2$ and J is either $\chi_1 J_1 \otimes J_2$ when $d \in \{2, 6\} \bmod 8$ or $J_1 \otimes J_2$ in other cases, see [33, 140].

Then $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a real commutative triple of dimension d such that $\tilde{A} \neq 0$ for some selfadjoint one-forms A , so is not exactly like in definition 4.3.1.

The vanishing tadpole of order 0 has the following equivalence (see [22])

$$\oint AD^{-1} = 0, \forall A \in \Omega_D^1(\mathcal{A}) \iff \oint ab = \oint a\alpha(b), \forall a, b \in \mathcal{A}, \quad (1.19)$$

where $\alpha(b) := \mathcal{D}b\mathcal{D}^{-1}$, equivalence which can be generalized as

Lemma 1.4.7. *In a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, for any $k \in \mathbb{N}$,*

$$\oint (AD^{-1})^n = 0, \forall A \in \Omega_D^1(\mathcal{A}), \forall n \in \{1, \dots, k\} \iff \oint \prod_{j=1}^k a_j \alpha(b_j) = \oint \prod_{j=1}^k a_j b_j, \forall a_j, b_j \in \mathcal{A}.$$

Proof. Note that $a[\mathcal{D}, b]\mathcal{D}^{-1} = a\tilde{\alpha}(b)$ where $\tilde{\alpha}(b) := \alpha(b) - b$.

Assuming the left hand-side, we get

$$\begin{aligned} 0 &= \oint (AD^{-1})^n = \oint a_1 \tilde{\alpha}(b_1) \dots a_j \tilde{\alpha}(b_j) \dots a_n \tilde{\alpha}(b_n) \\ &= \oint a_1 \tilde{\alpha}(b_1) \dots a_j \alpha(b_j) a_{j+1} \tilde{\alpha}(b_{j+1}) \dots a_k \tilde{\alpha}(b_k) - \oint a_1 \tilde{\alpha}(b_1) \dots a_j b_j a_{j+1} \tilde{\alpha}(b_{j+1}) \dots a_n \tilde{\alpha}(b_n) \end{aligned}$$

$\forall a_j, b_j \in \mathcal{A}$. But the last term is zero if $\oint (AD^{-1})^{n-1} = 0$ for all A . By induction, we end up with $0 = \oint a_1 \alpha(b_1) \dots a_{n-1} \alpha(b_{n-1}) a_n \tilde{\alpha}(b_n)$. Varying n between 1 and k , we get the right hand-side. \square

Chapitre 2

Spectral action on noncommutative torus

2.1 Introduction

In [60], the spectral action on NC-tori was only computed only for operators of the form $\mathcal{D} + A$ and computed for \mathcal{D}_A in [62]. It appears that the implementation of the real structure via J , does not change the spectral action, up to a coefficient, when the torus has dimension 4. Here we prove that this can be also directly obtained from the Chamseddine–Connes analysis of [22] that we follow quite closely. Let us recall that

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = \sum_{0 < k \in Sd^+} \Phi_k \Lambda^k \int |D_A|^{-k} + \Phi(0) \zeta_{D_A}(0) + \mathcal{O}(\Lambda^{-1})$$

where $D_A = \mathcal{D}_A + P_A$, P_A the projection on $\text{Ker } \mathcal{D}_A$, $\Phi_k = \frac{1}{2} \int_0^\infty \Phi(t) t^{k/2-1} dt$ and Sd^+ is the strictly positive part of the spectrum dimension of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.

In section 2, all previous technical points are widely used for the computation of terms in (1.6) or (1.11) on noncommutative torus. Most of the terms are based on residues of certain zeta functions and series of zeta functions that are studied in section 4. We show in particular that the vanishing tadpole hypothesis is satisfied on the torus.

The spectral action is obtained in section 3 and we conjecture that the noncommutative spectral action of \mathcal{D}_A has terms proportional to the spectral action of $\mathcal{D} + A$ on the commutative torus.

Since the computation of zeta functions is crucial here, we investigate in section 4 residues of series and integrals. This section contains independent interesting results on the holomorphy of series of holomorphic functions. In particular, the necessity of a Diophantine constraint is naturally emphasized.

All these results on spectral action are quite important in physics, especially in quantum field theory and particle physics, where one adds to the effective action some counterterms explicitly given by (1.11), see for instance [17, 21, 22, 24, 58, 60, 62, 66, 93, 135, 142–144].

2.2 The noncommutative torus

2.2.1 Notations

Let $C^\infty(\mathcal{T}_\Theta^n)$ be the smooth noncommutative n -torus associated to a non-zero skew-symmetric deformation matrix $\Theta \in M_n(\mathbb{R})$ (see [26], [115]). This means that $C^\infty(\mathcal{T}_\Theta^n)$ is the algebra generated by n unitaries u_i , $i = 1, \dots, n$ subject to the relations

$$u_i u_j = e^{i\Theta_{ij}} u_j u_i, \quad (2.1)$$

and with Schwartz coefficients: an element $a \in C^\infty(\mathcal{T}_\Theta^n)$ can be written as $a = \sum_{k \in \mathbb{Z}^n} a_k U_k$, where $\{a_k\} \in \mathcal{S}(\mathbb{Z}^n)$ with the Weyl elements defined by $U_k := e^{-\frac{i}{2}k \cdot \chi^k} u_1^{k_1} \dots u_n^{k_n}$, $k \in \mathbb{Z}^n$, relation (2.1) reads

$$U_k U_q = e^{-\frac{i}{2}k \cdot \Theta q} U_{k+q}, \text{ and } U_k U_q = e^{-ik \cdot \Theta q} U_q U_k \quad (2.2)$$

where χ is the matrix restriction of Θ to its upper triangular part. Thus unitary operators U_k satisfy $U_k^* = U_{-k}$ and $[U_k, U_l] = -2i \sin(\frac{1}{2}k \cdot \Theta l) U_{k+l}$.

Let τ be the trace on $C^\infty(\mathcal{T}_\Theta^n)$ defined by $\tau(\sum_{k \in \mathbb{Z}^n} a_k U_k) := a_0$ and \mathcal{H}_τ be the GNS Hilbert space obtained by completion of $C^\infty(\mathcal{T}_\Theta^n)$ with respect of the norm induced by the scalar product $\langle a, b \rangle := \tau(a^* b)$. On $\mathcal{H}_\tau = \{\sum_{k \in \mathbb{Z}^n} a_k U_k : \{a_k\}_k \in l^2(\mathbb{Z}^n)\}$, we consider the left and right regular representations of $C^\infty(\mathcal{T}_\Theta^n)$ by bounded operators, that we denote respectively by $L(\cdot)$ and $R(\cdot)$.

Let also δ_μ , $\mu \in \{1, \dots, n\}$, be the n (pairwise commuting) canonical derivations, defined by

$$\delta_\mu(U_k) := ik_\mu U_k. \quad (2.3)$$

We need to fix notations: let $\mathcal{A}_\Theta := C^\infty(\mathcal{T}_\Theta^n)$ acting on $\mathcal{H} := \mathcal{H}_\tau \otimes \mathbb{C}^{2^m}$ with $n = 2m$ or $n = 2m + 1$ (i.e., $m = \lfloor \frac{n}{2} \rfloor$ is the integer part of $\frac{n}{2}$), the square integrable sections of the trivial spin bundle over \mathcal{T}^n .

Each element of \mathcal{A}_Θ is represented on \mathcal{H} as $L(a) \otimes 1_{2^m}$ where L (resp. R) is the left (resp. right) multiplication. The Tomita conjugation $J_0(a) := a^*$ satisfies $[J_0, \delta_\mu] = 0$ and we define $J := J_0 \otimes C_0$ where C_0 is an operator on \mathbb{C}^{2^m} . The Dirac operator is given by

$$\mathcal{D} := -i \delta_\mu \otimes \gamma^\mu, \quad (2.4)$$

where we use hermitian Dirac matrices γ . It is defined and symmetric on the dense subset of \mathcal{H} given by $C^\infty(\mathcal{T}_\Theta^n) \otimes \mathbb{C}^{2^m}$. We still note \mathcal{D} its selfadjoint extension. This implies

$$C_0 \gamma^\alpha = -\varepsilon \gamma^\alpha C_0, \quad (2.5)$$

and

$$\mathcal{D} U_k \otimes e_i = k_\mu U_k \otimes \gamma^\mu e_i,$$

where (e_i) is the canonical basis of \mathbb{C}^{2^m} . Moreover, $C_0^2 = \pm 1_{2^m}$ depending on the parity of m . Finally, one introduces the chirality (which in the even case is $\chi := id \otimes (-i)^m \gamma^1 \dots \gamma^n$) and this yields that $(\mathcal{A}_\Theta, \mathcal{H}, \mathcal{D}, J, \chi)$ satisfies all axioms of a spectral triple, see [28, 68].

The perturbed Dirac operator $V_u \mathcal{D} V_u^*$ by the unitary

$$V_u := (L(u) \otimes 1_{2^m}) J (L(u) \otimes 1_{2^m}) J^{-1},$$

defined for every unitary $u \in \mathcal{A}$, $uu^* = u^*u = U_0$, must satisfy condition (1.3) (which is equivalent to \mathcal{H} being endowed with a structure of \mathcal{A}_Θ -bimodule). This yields the necessity of a symmetrized covariant Dirac operator:

$$\mathcal{D}_A := \mathcal{D} + A + \epsilon J A J^{-1}$$

since $V_u \mathcal{D} V_u^* = \mathcal{D}_{L(u) \otimes 1_{2^m} [\mathcal{D}, L(u^*) \otimes 1_{2^m}]}$: in fact, for $a \in \mathcal{A}_\Theta$, using $J_0 L(a) J_0^{-1} = R(a^*)$, we get

$$\epsilon J (L(a) \otimes \gamma^\alpha) J^{-1} = -R(a^*) \otimes \gamma^\alpha$$

and that the representation L and the anti-representation R are \mathbb{C} -linear, commute and satisfy

$$[\delta_\alpha, L(a)] = L(\delta_\alpha a), \quad [\delta_\alpha, R(a)] = R(\delta_\alpha a).$$

This induces some covariance property for the Dirac operator: one checks that for all $k \in \mathbb{Z}^n$,

$$L(U_k) \otimes 1_{2^m} [\mathcal{D}, L(U_k^*) \otimes 1_{2^m}] = 1 \otimes (-k_\mu \gamma^\mu), \quad (2.6)$$

so with (2.5), we get $U_k [\mathcal{D}, U_k^*] + \epsilon J U_k [\mathcal{D}, U_k^*] J^{-1} = 0$ and

$$V_{U_k} \mathcal{D} V_{U_k}^* = \mathcal{D} = \mathcal{D}_{L(U_k) \otimes 1_{2^m} [\mathcal{D}, L(U_k^*) \otimes 1_{2^m}]}. \quad (2.7)$$

Moreover, we get the gauge transformation:

$$V_u \mathcal{D}_A V_u^* = \mathcal{D}_{\gamma_u(A)} \quad (2.8)$$

where the gauged transform one-form of A is

$$\gamma_u(A) := u[\mathcal{D}, u^*] + u A u^*, \quad (2.9)$$

with the shorthand $L(u) \otimes 1_{2^m} \longrightarrow u$.

As a consequence, the spectral action is gauge invariant:

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = \mathcal{S}(\mathcal{D}_{\gamma_u(A)}, \Phi, \Lambda).$$

An arbitrary selfadjoint one-form A , can be written as

$$A = L(-i A_\alpha) \otimes \gamma^\alpha, \quad A_\alpha = -A_\alpha^* \in \mathcal{A}_\Theta, \quad (2.10)$$

thus

$$\mathcal{D}_A = -i (\delta_\alpha + L(A_\alpha) - R(A_\alpha)) \otimes \gamma^\alpha. \quad (2.11)$$

Defining

$$\tilde{A}_\alpha := L(A_\alpha) - R(A_\alpha),$$

we get $\mathcal{D}_A^2 = -g^{\alpha_1 \alpha_2} (\delta_{\alpha_1} + \tilde{A}_{\alpha_1}) (\delta_{\alpha_2} + \tilde{A}_{\alpha_2}) \otimes 1_{2^m} - \frac{1}{2} \Omega_{\alpha_1 \alpha_2} \otimes \gamma^{\alpha_1 \alpha_2}$ where

$$\gamma^{\alpha_1 \alpha_2} := \frac{1}{2} (\gamma^{\alpha_1} \gamma^{\alpha_2} - \gamma^{\alpha_2} \gamma^{\alpha_1}),$$

$$\Omega_{\alpha_1 \alpha_2} := [\delta_{\alpha_1} + \tilde{A}_{\alpha_1}, \delta_{\alpha_2} + \tilde{A}_{\alpha_2}] = L(F_{\alpha_1 \alpha_2}) - R(F_{\alpha_1 \alpha_2})$$

with

$$F_{\alpha_1 \alpha_2} := \delta_{\alpha_1}(A_{\alpha_2}) - \delta_{\alpha_2}(A_{\alpha_1}) + [A_{\alpha_1}, A_{\alpha_2}]. \quad (2.12)$$

In summary,

$$\begin{aligned} \mathcal{D}_A^2 = & -g^{\alpha_1 \alpha_2} \left(\delta_{\alpha_1} + L(A_{\alpha_1}) - R(A_{\alpha_1}) \right) \left(\delta_{\alpha_2} + L(A_{\alpha_2}) - R(A_{\alpha_2}) \right) \otimes 1_{2^m} \\ & - \frac{1}{2} (L(F_{\alpha_1 \alpha_2}) - R(F_{\alpha_1 \alpha_2})) \otimes \gamma^{\alpha_1 \alpha_2}. \end{aligned} \quad (2.13)$$

2.2.2 Kernels and dimension spectrum

We now compute the kernel of the perturbed Dirac operator:

Proposition 2.2.1. (i) $\text{Ker } \mathcal{D} = U_0 \otimes \mathbb{C}^{2^m}$, so $\dim \text{Ker } \mathcal{D} = 2^m$.

(ii) For any selfadjoint one-form A , $\text{Ker } \mathcal{D} \subseteq \text{Ker } \mathcal{D}_A$.

(iii) For any unitary $u \in \mathcal{A}$, $\text{Ker } \mathcal{D}_{\gamma_u(A)} = V_u \text{Ker } \mathcal{D}_A$.

Proof. (i) Let $\psi = \sum_{k,j} c_{k,j} U_k \otimes e_j \in \text{Ker } \mathcal{D}$. Thus, $0 = \mathcal{D}^2 \psi = \sum_{k,i} c_{k,j} |k|^2 U_k \otimes e_j$ which entails that $c_{k,j} |k|^2 = 0$ for any $k \in \mathbb{Z}^n$ and $1 \leq j \leq 2^m$. The result follows.

(ii) Let $\psi \in \text{Ker } \mathcal{D}$. So, $\psi = U_0 \otimes v$ with $v \in \mathbb{C}^{2^m}$ and from (2.11), we get

$$\mathcal{D}_A \psi = \mathcal{D} \psi + (A + \epsilon J A J^{-1}) \psi = (A + \epsilon J A J^{-1}) \psi = -i[A_\alpha, U_0] \otimes \gamma^\alpha v = 0$$

since U_0 is the unit of the algebra, which proves that $\psi \in \text{Ker } \mathcal{D}_A$.

(iii) This is a direct consequence of (2.8). \square

Corollary 2.2.2. Let A be a selfadjoint one-form. Then $\text{Ker } \mathcal{D}_A = \text{Ker } \mathcal{D}$ in the following cases:

(i) $A_u := L(u) \otimes 1_{2^m} [\mathcal{D}, L(u^*) \otimes 1_{2^m}]$ when u is a unitary in \mathcal{A} .

(ii) $\|A\| < \frac{1}{2}$.

(iii) The matrix $\frac{1}{2\pi} \Theta$ has only integral coefficients.

Proof. (i) This follows from previous result because $V_u(U_0 \otimes v) = U_0 \otimes v$ for any $v \in \mathbb{C}^{2^m}$.

(ii) Let $\psi = \sum_{k,j} c_{k,j} U_k \otimes e_j$ be in $\text{Ker } \mathcal{D}_A$ (so $\sum_{k,j} |c_{k,j}|^2 < \infty$) and $\phi := \sum_j c_{0,j} U_0 \otimes e_j$. Thus $\psi' := \psi - \phi \in \text{Ker } \mathcal{D}_A$ since $\phi \in \text{Ker } \mathcal{D} \subseteq \text{Ker } \mathcal{D}_A$ and

$$\| \sum_{0 \neq k \in \mathbb{Z}^n, j} c_{k,j} k_\alpha U_k \otimes \gamma^\alpha e_j \|^2 = \|\mathcal{D} \psi'\|^2 = \|-(A + \epsilon J A J^{-1}) \psi'\|^2 \leq 4\|A\|^2 \|\psi'\|^2 < \|\psi'\|^2.$$

Defining $X_k := \sum_\alpha k_\alpha \gamma_\alpha$, $X_k^2 = \sum_\alpha |k_\alpha|^2 1_{2^m}$ is invertible and the vectors $\{U_k \otimes X_k e_j\}_{0 \neq k \in \mathbb{Z}^n, j}$ are orthogonal in \mathcal{H} , so

$$\sum_{0 \neq k \in \mathbb{Z}^n, j} \left(\sum_\alpha |k_\alpha|^2 \right) |c_{k,j}|^2 < \sum_{0 \neq k \in \mathbb{Z}^n, j} |c_{k,j}|^2$$

which is possible only if $c_{k,j} = 0$, $\forall k, j$ that is $\psi' = 0$ et $\psi = \phi \in \text{Ker } \mathcal{D}$.

(iii) This is a consequence of the fact that the algebra is commutative, thus $A + \epsilon J A J^{-1} = 0$. \square

Note that if $\tilde{A}_u := A_u + \epsilon J A_u J^{-1}$, then by (2.6), $\tilde{A}_{U_k} = 0$ for all $k \in \mathbb{Z}^n$ and $\|A_{U_k}\| = |k|$, but for an arbitrary unitary $u \in \mathcal{A}$, $\tilde{A}_u \neq 0$ so $\mathcal{D}_{A_u} \neq \mathcal{D}$.

Naturally the above result is also a direct consequence of the fact that the eigenspace of an isolated eigenvalue of an operator is not modified by small perturbations. However, it is interesting to compute the last result directly to emphasize the difficulty of the general case:

Let $\psi = \sum_{l \in \mathbb{Z}^n, 1 \leq j \leq 2^m} c_{l,j} U_l \otimes e_j \in \text{Ker } \mathcal{D}_A$, so $\sum_{l \in \mathbb{Z}^n, 1 \leq j \leq 2^m} |c_{l,j}|^2 < \infty$. We have to show that $\psi \in \text{Ker } \mathcal{D}$ that is $c_{l,j} = 0$ when $l \neq 0$.

Taking the scalar product of $\langle U_k \otimes e_i |$ with

$$0 = \mathcal{D}_A \psi = \sum_{l, \alpha, j} c_{l,j} (l^\alpha U_l - i[A_\alpha, U_l]) \otimes \gamma^\alpha e_j,$$

we obtain

$$0 = \sum_{l, \alpha, j} c_{l, j} (l^\alpha \delta_{k, l} - i \langle U_k, [A_\alpha, U_l] \rangle) \langle e_i, \gamma^\alpha e_j \rangle.$$

If $A_\alpha = \sum_{\alpha, l} a_{\alpha, l} U_l \otimes \gamma^\alpha$ with $\{a_{\alpha, l}\}_l \in \mathcal{S}(\mathbb{Z}^n)$, note that $[U_l, U_m] = -2i \sin(\frac{1}{2}l \cdot \Theta m) U_{l+m}$ and

$$\langle U_k, [A_\alpha, U_l] \rangle = \sum_{l' \in \mathbb{Z}^n} a_{\alpha, l'} (-2i \sin(\frac{1}{2}l' \cdot \Theta l)) \langle U_k, U_{l'+l} \rangle = -2i a_{\alpha, k-l} \sin(\frac{1}{2}k \cdot \Theta l).$$

Thus

$$0 = \sum_{l \in \mathbb{Z}^n} \sum_{\alpha=1}^n \sum_{j=1}^{2^m} c_{l, j} (l^\alpha \delta_{k, l} - 2a_{\alpha, k-l} \sin(\frac{1}{2}k \cdot \Theta l)) \langle e_i, \gamma^\alpha e_j \rangle, \quad \forall k \in \mathbb{Z}^n, \forall i, 1 \leq i \leq 2^m. \quad (2.14)$$

We conjecture that $\text{Ker } \mathcal{D} = \text{Ker } \mathcal{D}_A$ at least for generic Θ 's:

the constraints (2.14) should imply $c_{l, j} = 0$ for all j and all $l \neq 0$ meaning $\psi \in \text{Ker } \mathcal{D}$. When $\frac{1}{2\pi}\Theta$ has only integer coefficients, the sin part of these constraints disappears giving the result. We shall use the following

Definition 2.2.3. (i) Let $\delta > 0$. A vector $a \in \mathbb{R}^n$ is said to be δ -diophantine if there exists $c > 0$ such that $|q \cdot a - m| \geq c |q|^{-\delta}$, $\forall q \in \mathbb{Z}^n \setminus \{0\}$ and $\forall m \in \mathbb{Z}$.

We note $\mathcal{BV}(\delta)$ the set of δ -diophantine vectors and $\mathcal{BV} := \cup_{\delta > 0} \mathcal{BV}(\delta)$ the set of diophantine vectors.

(ii) A matrix $\Theta \in \mathcal{M}_n(\mathbb{R})$ (real $n \times n$ matrices) will be said to be diophantine if there exists $u \in \mathbb{Z}^n$ such that ${}^t\Theta(u)$ is a diophantine vector of \mathbb{R}^n .

Lemma 2.2.4. If $\frac{1}{2\pi}\Theta$ is diophantine, $Sp(C^\infty(\mathcal{T}_\Theta), \mathcal{H}, \mathcal{D}) = \mathbb{Z}$ and all these poles are simple.

Proof. Let $B \in \mathcal{D}(\mathcal{A})$ and $p \in \mathbb{N}_0$. Suppose that B is of the form

$$B = a_r b_r \mathcal{D}^{q_r-1} |\mathcal{D}|^{p_r-1} a_{r-1} b_{r-1} \cdots \mathcal{D}^{q_1} |\mathcal{D}|^{p_1} a_1 b_1$$

where $r \in \mathbb{N}$, $a_i \in \mathcal{A}$, $b_i \in JAJ^{-1}$, $q_i, p_i \in \mathbb{N}_0$. We note $a_i =: \sum_l a_{i, l} U_l$ and $b_i =: \sum_l b_{i, l} U_l$. With the shorthand $k_{\mu_1, \mu_{q_i}} := k_{\mu_1} \cdots k_{\mu_{q_i}}$ and $\gamma^{\mu_1, \mu_{q_i}} = \gamma^{\mu_1} \cdots \gamma^{\mu_{q_i}}$, we get

$$\mathcal{D}^{q_1} |\mathcal{D}|^{p_1} a_1 b_1 U_k \otimes e_j = \sum_{l_1, l'_1} a_{1, l_1} b_{1, l'_1} U_{l_1} U_k U_{l'_1} |k + l_1 + l'_1|^{p_1} (k + l_1 + l'_1)_{\mu_1, \mu_{q_1}} \otimes \gamma^{\mu_1, \mu_{q_1}} e_j$$

which gives, after r iterations,

$$BU_k \otimes e_j = \sum_{l, l'} \tilde{a}_l \tilde{b}_{l'} U_{l_r} \cdots U_{l_1} U_k U_{l'_1} \cdots U_{l'_r} \prod_{i=1}^{r-1} |k + \widehat{l}_i + \widehat{l}'_i|^{p_i} (k + \widehat{l}_i + \widehat{l}'_i)_{\mu_1^i, \mu_{q_i}^i} \otimes \gamma^{\mu_1^{r-1}, \mu_{q_{r-1}}^{r-1}} \cdots \gamma^{\mu_1^1, \mu_{q_1}^1} e_j$$

where $\tilde{a}_l := a_{1, l_1} \cdots a_{r, l_r}$ and $\tilde{b}_{l'} := b_{1, l'_1} \cdots b_{r, l'_r}$.

Let us note $F_\mu(k, l, l') := \prod_{i=1}^{r-1} |k + \widehat{l}_i + \widehat{l}'_i|^{p_i} (k + \widehat{l}_i + \widehat{l}'_i)_{\mu_1^i, \mu_{q_i}^i}$ and $\gamma^\mu := \gamma^{\mu_1^{r-1}, \mu_{q_{r-1}}^{r-1}} \cdots \gamma^{\mu_1^1, \mu_{q_1}^1}$. Thus, with the shorthand \sim_c meaning modulo a constant function towards the variable s ,

$$\text{Tr}(B|\mathcal{D}|^{-2p-s}) \sim_c \sum_k' \sum_{l, l'} \tilde{a}_l \tilde{b}_{l'} \tau(U_{-k} U_{l_r} \cdots U_{l_1} U_k U_{l'_1} \cdots U_{l'_r}) \frac{F_\mu(k, l, l')}{|k|^{s+2p}} \text{Tr}(\gamma^\mu).$$

Since $U_{l_r} \cdots U_{l_1} U_k = U_k U_{l_r} \cdots U_{l_1} e^{-i \sum_1^r l_i \cdot \Theta k}$ we get

$$\tau(U_{-k} U_{l_r} \cdots U_{l_1} U_k U_{l'_1} \cdots U_{l'_r}) = \delta_{\sum_1^r l_i + l'_i, 0} e^{i\phi(l, l')} e^{-i \sum_1^r l_i \cdot \Theta k}$$

where ϕ is a real valued function. Thus,

$$\begin{aligned} \text{Tr}(B|D|^{-2p-s}) &\sim_c \sum_k' \sum_{l, l'} e^{i\phi(l, l')} \delta_{\sum_1^r l_i + l'_i, 0} \tilde{a}_l \tilde{b}_{l'} \frac{F_\mu(k, l, l') e^{-i \sum_1^r l_i \cdot \Theta k}}{|k|^{s+2p}} \text{Tr}(\gamma^\mu) \\ &\sim_c f_\mu(s) \text{Tr}(\gamma^\mu). \end{aligned}$$

The function $f_\mu(s)$ can be decomposed has a linear combination of zeta function of type described in Theorem 2.4.16 (or, if $r = 1$ or all the p_i are zero, in Theorem 2.4.4). Thus, $s \mapsto \text{Tr}(B|D|^{-2p-s})$ has only poles in \mathbb{Z} and each pole is simple. Finally, by linearity, we get the result. \square

The dimension spectrum of the noncommutative torus is simple:

Proposition 2.2.5. (i) If $\frac{1}{2\pi}\Theta$ is diophantine, the spectrum dimension of $(C^\infty(\mathcal{T}_\Theta^n), \mathcal{H}, \mathcal{D})$ is equal to the set $\{n - k : k \in \mathbb{N}_0\}$ and all these poles are simple.

(ii) $\zeta_D(0) = 0$.

Proof. (i) Lemma 2.2.4 and Remark 1.2.11.

(ii) $\zeta_D(s) = \sum_{k \in \mathbb{Z}^n} \sum_{1 \leq j \leq 2^m} \langle U_k \otimes e_j, |D|^{-s} U_k \otimes e_j \rangle = 2^m (\sum_{k \in \mathbb{Z}^n}' \frac{1}{|k|^s} + 1) = 2^m (Z_n(s) + 1)$. By (2.42), we get the result. \square

We have computed $\zeta_D(0)$ relatively easily but the main difficulty of the present work is precisely to calculate $\zeta_{D_A}(0)$.

2.2.3 Noncommutative integral computations

We fix a self-adjoint 1-form A on the noncommutative torus of dimension n .

Proposition 2.2.6. If $\frac{1}{2\pi}\Theta$ is diophantine, then the first elements of the expansion (1.6) are given by

$$\begin{aligned} \oint |D_A|^{-n} &= \oint |D|^{-n} = 2^{m+1} \pi^{n/2} \Gamma(\frac{n}{2})^{-1}. \\ \oint |D_A|^{n-k} &= 0 \text{ for } k \text{ odd.} \\ \oint |D_A|^{n-2} &= 0. \end{aligned} \tag{2.15}$$

We need few technical lemmas:

Lemma 2.2.7. On the noncommutative torus, for any $t \in \mathbb{R}$,

$$\oint \tilde{A} \mathcal{D} |D|^{-t} = \oint \mathcal{D} \tilde{A} |D|^{-t} = 0.$$

Proof. Using notations of (2.10), we have

$$\begin{aligned} \mathrm{Tr}(\tilde{\mathcal{A}}\mathcal{D}|D|^{-s}) &\sim_c \sum_j \sum_k' \langle U_k \otimes e_j, -ik_\mu |k|^{-s} [A_\alpha, U_k] \otimes \gamma^\alpha \gamma^\mu e_j \rangle \\ &\sim_c -i \mathrm{Tr}(\gamma^\alpha \gamma^\mu) \sum_k' k_\mu |k|^{-s} \langle U_k, [A_\alpha, U_k] \rangle = 0 \end{aligned}$$

since $\langle U_k, [A_\alpha, U_k] \rangle = 0$. Similarly

$$\begin{aligned} \mathrm{Tr}(\mathcal{D}\tilde{\mathcal{A}}|D|^{-s}) &\sim_c \sum_j \sum_k' \langle U_k \otimes e_j, |k|^{-s} \sum_l a_{\alpha,l} 2 \sin \frac{k \cdot \Theta l}{2} (l+k)_\mu U_{l+k} \otimes \gamma^\mu \gamma^\alpha e_j \rangle \\ &\sim_c 2 \mathrm{Tr}(\gamma^\mu \gamma^\alpha) \sum_k' \sum_l a_{\alpha,l} \sin \frac{k \cdot \Theta l}{2} (l+k)_\mu |k|^{-s} \langle U_k, U_{l+k} \rangle = 0. \quad \square \end{aligned}$$

Any element h in the algebra generated by \mathcal{A} and $[\mathcal{D}, \mathcal{A}]$ can be written as a linear combination of terms of the form $a_1^{p_1} \cdots a_n^{p_r}$ where a_i are elements of \mathcal{A} or $[\mathcal{D}, \mathcal{A}]$. Such a term can be written as a series $b := \sum a_{1,\alpha_1,l_1} \cdots a_{q,\alpha_q,l_q} U_{l_1} \cdots U_{l_q} \otimes \gamma^{\alpha_1} \cdots \gamma^{\alpha_q}$ where a_{i,α_i} are Schwartz sequences and when $a_i = \sum_l a_l U_l \in \mathcal{A}$, we set $a_{i,\alpha,l} = a_{i,l}$ with $\gamma^\alpha = 1$. We define

$$L(b) := \tau \left(\sum_l a_{1,\alpha_1,l_1} \cdots a_{q,\alpha_q,l_q} U_{l_1} \cdots U_{l_q} \right) \mathrm{Tr}(\gamma^{\alpha_1} \cdots \gamma^{\alpha_q}).$$

By linearity, L is defined as a linear form on the whole algebra generated by \mathcal{A} and $[\mathcal{D}, \mathcal{A}]$.

Lemma 2.2.8. *If h is an element of the algebra generated by \mathcal{A} and $[\mathcal{D}, \mathcal{A}]$,*

$$\mathrm{Tr}(h|D|^{-s}) \sim_c L(h) Z_n(s).$$

In particular, $\mathrm{Tr}(h|D|^{-s})$ has at most one pole at $s = n$.

Proof. We get with b of the form $\sum a_{1,\alpha_1,l_1} \cdots a_{q,\alpha_q,l_q} U_{l_1} \cdots U_{l_q} \otimes \gamma^{\alpha_1} \cdots \gamma^{\alpha_q}$,

$$\begin{aligned} \mathrm{Tr}(b|D|^{-s}) &\sim_c \sum_{k \in \mathbb{Z}^n}' \langle U_k, \sum_l a_{1,\alpha_1,l_1} \cdots a_{q,\alpha_q,l_q} U_{l_1} \cdots U_{l_q} U_k \rangle \mathrm{Tr}(\gamma^{\alpha_1} \cdots \gamma^{\alpha_q}) |k|^{-s} \\ &\sim_c \tau \left(\sum_l a_{1,\alpha_1,l_1} \cdots a_{q,\alpha_q,l_q} U_{l_1} \cdots U_{l_q} \right) \mathrm{Tr}(\gamma^{\alpha_1} \cdots \gamma^{\alpha_q}) Z_n(s) = L(b) Z_n(s). \end{aligned}$$

The results follows now from linearity of the trace. \square

Lemma 2.2.9. *If $\frac{1}{2\pi}\Theta$ is diophantine, the function $s \mapsto \mathrm{Tr}(\varepsilon JAJ^{-1}A|D|^{-s})$ extends meromorphically on the whole plane with only one possible pole at $s = n$. Moreover, this pole is simple and*

$$\mathrm{Res}_{s=n} \mathrm{Tr}(\varepsilon JAJ^{-1}A|D|^{-s}) = a_{\alpha,0} a_0^\alpha 2^{m+1} \pi^{n/2} \Gamma(n/2)^{-1}.$$

Proof. With $A = L(-iA_\alpha) \otimes \gamma^\alpha$, we get $\varepsilon JAJ^{-1} = R(iA_\alpha) \otimes \gamma^\alpha$, and by multiplication $\varepsilon JAJ^{-1}A = R(A_\beta)L(A_\alpha) \otimes \gamma^\beta \gamma^\alpha$. Thus,

$$\begin{aligned} \mathrm{Tr}(\varepsilon JAJ^{-1}A|D|^{-s}) &\sim_c \sum_{k \in \mathbb{Z}^n}' \langle U_k, A_\alpha U_k A_\beta \rangle |k|^{-s} \mathrm{Tr}(\gamma^\beta \gamma^\alpha) \\ &\sim_c \sum_{k \in \mathbb{Z}^n}' \sum_l a_{\alpha,l} a_{\beta,-l} e^{ik \cdot \Theta l} |k|^{-s} \mathrm{Tr}(\gamma^\beta \gamma^\alpha) \\ &\sim_c 2^m \sum_{k \in \mathbb{Z}^n}' \sum_l a_{\alpha,l} a_{-l}^\alpha e^{ik \cdot \Theta l} |k|^{-s}. \end{aligned}$$

Theorem 2.4.4 (ii) entails that $\sum'_{k \in \mathbb{Z}^n} \sum_l a_{\alpha,l} a_{-l}^\alpha e^{ik \cdot \Theta l} |k|^{-s}$ extends meromorphically on the whole plane \mathbb{C} with only one possible pole at $s = n$. Moreover, this pole is simple and we have

$$\text{Res}_{s=n} \sum'_{k \in \mathbb{Z}^n} \sum_l a_{\alpha,l} a_{-l}^\alpha e^{ik \cdot \Theta l} |k|^{-s} = a_{\alpha,0} a_0^\alpha \text{Res}_{s=n} Z_n(s).$$

Equation (2.41) now gives the result. \square

Lemma 2.2.10. *If $\frac{1}{2\pi}\Theta$ is diophantine, then for any $t \in \mathbb{R}$,*

$$\oint X |D|^{-t} = \delta_{t,n} 2^{m+1} \left(- \sum_l a_{\alpha,l} a_{-l}^\alpha + a_{\alpha,0} a_0^\alpha \right) 2\pi^{n/2} \Gamma(n/2)^{-1}.$$

where $X = \tilde{A}\mathcal{D} + \mathcal{D}\tilde{A} + \tilde{A}^2$ and $A =: -i \sum_l a_{\alpha,l} U_l \otimes \gamma^\alpha$.

Proof. By Lemma 2.2.7, we get $\oint X |D|^{-t} = \text{Res}_{s=0} \text{Tr}(\tilde{A}^2 |D|^{-s-t})$. Since A and $\varepsilon J A J^{-1}$ commute, we have $\tilde{A}^2 = A^2 + J A^2 J^{-1} + 2\varepsilon J A J^{-1} A$. Thus,

$$\text{Tr}(\tilde{A}^2 |D|^{-s-t}) = \text{Tr}(A^2 |D|^{-s-t}) + \text{Tr}(J A^2 J^{-1} |D|^{-s-t}) + 2 \text{Tr}(\varepsilon J A J^{-1} A |D|^{-s-t}).$$

Since $|D|$ and J commute, we have with Lemma 2.2.8,

$$\text{Tr}(\tilde{A}^2 |D|^{-s-t}) \sim_c 2L(A^2) Z_n(s+t) + 2 \text{Tr}(\varepsilon J A J^{-1} A |D|^{-s-t}).$$

Thus Lemma 2.2.9 entails that $\text{Tr}(\tilde{A}^2 |D|^{-s-t})$ is holomorphic at 0 if $t \neq n$. When $t = n$,

$$\text{Res}_{s=0} \text{Tr}(\tilde{A}^2 |D|^{-s-t}) = 2^{m+1} \left(- \sum_l a_{\alpha,l} a_{-l}^\alpha + a_{\alpha,0} a_0^\alpha \right) 2\pi^{n/2} \Gamma(n/2)^{-1}, \quad (2.16)$$

which gives the result. \square

Lemma 2.2.11. *If $\frac{1}{2\pi}\Theta$ is diophantine, then*

$$\oint \tilde{A} \mathcal{D} \tilde{A} \mathcal{D} |D|^{-2-n} = -\frac{n-2}{n} \oint \tilde{A}^2 |D|^{-n}.$$

Proof. With $\mathcal{D}J = \varepsilon J \mathcal{D}$, we get

$$\oint \tilde{A} \mathcal{D} \tilde{A} \mathcal{D} |D|^{-2-n} = 2 \oint A \mathcal{D} A \mathcal{D} |D|^{-2-n} + 2 \oint \varepsilon J A J^{-1} \mathcal{D} A \mathcal{D} |D|^{-2-n}.$$

Let us first compute $\oint A \mathcal{D} A \mathcal{D} |D|^{-2-n}$. We have, with $A =: -iL(A_\alpha) \otimes \gamma^\alpha =: -i \sum_l a_{\alpha,l} U_l \otimes \gamma^\alpha$,

$$\text{Tr}(A \mathcal{D} A \mathcal{D} |D|^{-s-2-n}) \sim_c - \sum_k' \sum_{l_1, l_2} a_{\alpha_2, l_2} a_{\alpha_1, l_1} \tau(U_{-k} U_{l_2} U_{l_1} U_k) \frac{k_{\mu_1} (k+l_1)_{\mu_2}}{|k|^{s+2+n}} \text{Tr}(\gamma^{\alpha, \mu})$$

where $\gamma^{\alpha, \mu} := \gamma^{\alpha_2} \gamma^{\mu_2} \gamma^{\alpha_1} \gamma^{\mu_1}$. Thus,

$$\oint A \mathcal{D} A \mathcal{D} |D|^{-2-n} = - \sum_l a_{\alpha_2, -l} a_{\alpha_1, l} \text{Res}_{s=0} \left(\sum_k' \frac{k_{\mu_1} k_{\mu_2}}{|k|^{s+2+n}} \right) \text{Tr}(\gamma^{\alpha, \mu}).$$

We have also, with $\varepsilon JAJ^{-1} = iR(A_\alpha) \otimes \gamma^\alpha$,

$$\mathrm{Tr}(\varepsilon JAJ^{-1} \mathcal{D}AD|D|^{-s-2-n}) \sim_c \sum_k' \sum_{l_1, l_2} a_{\alpha_2, l_2} a_{\alpha_1, l_1} \tau(U_{-k} U_{l_1} U_k U_{l_2}) \frac{k_{\mu_1} (k+l_1)_{\mu_2}}{|k|^{s+2+n}} \mathrm{Tr}(\gamma^{\alpha, \mu}).$$

which gives

$$\oint \varepsilon JAJ^{-1} \mathcal{D}AD|D|^{-2-n} = a_{\alpha_2, 0} a_{\alpha_1, 0} \mathrm{Res}_{s=0} \left(\sum_k' \frac{k_{\mu_1} k_{\mu_2}}{|k|^{s+2+n}} \right) \mathrm{Tr}(\gamma^{\alpha, \mu}).$$

Thus,

$$\frac{1}{2} \oint \tilde{A} \mathcal{D} \tilde{A} \mathcal{D} |D|^{-2-n} = (a_{\alpha_2, 0} a_{\alpha_1, 0} - \sum_l a_{\alpha_2, -l} a_{\alpha_1, l}) \mathrm{Res}_{s=0} \left(\sum_k' \frac{k_{\mu_1} k_{\mu_2}}{|k|^{s+2+n}} \right) \mathrm{Tr}(\gamma^{\alpha, \mu}).$$

With $\sum_k' \frac{k_{\mu_1} k_{\mu_2}}{|k|^{s+2+n}} = \frac{\delta_{\mu_1 \mu_2}}{n} Z_n(s+n)$ and $C_n := \mathrm{Res}_{s=0} Z_n(s+n) = 2\pi^{n/2} \Gamma(n/2)^{-1}$ we obtain

$$\frac{1}{2} \oint \tilde{A} \mathcal{D} \tilde{A} \mathcal{D} |D|^{-2-n} = (a_{\alpha_2, 0} a_{\alpha_1, 0} - \sum_l a_{\alpha_2, -l} a_{\alpha_1, l}) \frac{C_n}{n} \mathrm{Tr}(\gamma^{\alpha_2} \gamma^\mu \gamma^{\alpha_1} \gamma_\mu).$$

Since $\mathrm{Tr}(\gamma^{\alpha_2} \gamma^\mu \gamma^{\alpha_1} \gamma_\mu) = 2^m (2-n) \delta^{\alpha_2, \alpha_1}$, we get

$$\frac{1}{2} \oint \tilde{A} \mathcal{D} \tilde{A} \mathcal{D} |D|^{-2-n} = 2^m \left(-a_{\alpha, 0} a_0^\alpha + \sum_l a_{\alpha, -l} a_l^\alpha \right) \frac{C_n (n-2)}{n}.$$

Lemma 2.2.10 now proves the result. \square

Lemma 2.2.12. *If $\frac{1}{2\pi} \Theta$ is diophantine, then for any $P \in \Psi_1(\mathcal{A})$ and $q \in \mathbb{N}$, q odd,*

$$\oint P |D|^{-(n-q)} = 0.$$

Proof. There exist $B \in \mathcal{D}_1(\mathcal{A})$ and $p \in \mathbb{N}_0$ such that $P = BD^{-2p} + R$ where R is in OP^{-q-1} . As a consequence, $\oint P |D|^{-(n-q)} = \oint B |D|^{-n-2p+q}$. Assume $B = a_r b_r \mathcal{D}^{q_r-1} a_{r-1} b_{r-1} \cdots \mathcal{D}^{q_1} a_1 b_1$ where $r \in \mathbb{N}$, $a_i \in \mathcal{A}$, $b_i \in JAJ^{-1}$, $q_i \in \mathbb{N}$. If we prove that $\oint B |D|^{-n-2p+q} = 0$, then the general case will follow by linearity. We note $a_i =: \sum_l a_{i,l} U_l$ and $b_i =: \sum_l b_{i,l} U_l$. With the shorthand $k_{\mu_1, \mu_{q_i}} := k_{\mu_1} \cdots k_{\mu_{q_i}}$ and $\gamma^{\mu_1, \mu_{q_i}} = \gamma^{\mu_1} \cdots \gamma^{\mu_{q_i}}$, we get

$$\mathcal{D}^{q_1} a_1 b_1 U_k \otimes e_j = \sum_{l_1, l'_1} a_{1, l_1} b_{1, l'_1} U_{l_1} U_k U_{l'_1} (k + l_1 + l'_1)_{\mu_1, \mu_{q_1}} \otimes \gamma^{\mu_1, \mu_{q_1}} e_j$$

which gives, after iteration,

$$B U_k \otimes e_j = \sum_{l, l'} \tilde{a}_l \tilde{b}_{l'} U_{l_r} \cdots U_{l_1} U_k U_{l'_1} \cdots U_{l'_r} \prod_{i=1}^{r-1} (k + \hat{l}_i + \hat{l}'_i)_{\mu_1^i, \mu_{q_i}^i} \otimes \gamma^{\mu_1^{r-1}, \mu_{q_{r-1}}^{r-1}} \cdots \gamma^{\mu_1^1, \mu_{q_1}^1} e_j$$

where $\tilde{a}_l := a_{1, l_1} \cdots a_{r, l_r}$ and $\tilde{b}_{l'} := b_{1, l'_1} \cdots b_{r, l'_r}$. Let's note $Q_\mu(k, l, l') := \prod_{i=1}^{r-1} (k + \hat{l}_i + \hat{l}'_i)_{\mu_1^i, \mu_{q_i}^i}$ and $\gamma^\mu := \gamma^{\mu_1^{r-1}, \mu_{q_{r-1}}^{r-1}} \cdots \gamma^{\mu_1^1, \mu_{q_1}^1}$. Thus,

$$\oint B |D|^{-n-2p+q} = \mathrm{Res}_{s=0} \sum_k' \sum_{l, l'} \tilde{a}_l \tilde{b}_{l'} \tau(U_{-k} U_{l_r} \cdots U_{l_1} U_k U_{l'_1} \cdots U_{l'_r}) \frac{Q_\mu(k, l, l')}{|k|^{s+2p+n-q}} \mathrm{Tr}(\gamma^\mu).$$

Since $U_{l_r} \cdots U_{l_1} U_k = U_k U_{l_r} \cdots U_{l_1} e^{-i \sum_1^r l_i \cdot \Theta k}$, we get

$$\tau(U_{-k} U_{l_r} \cdots U_{l_1} U_k U_{l'_1} \cdots U_{l'_r}) = \delta_{\sum_1^r l_i + l'_i, 0} e^{i\phi(l, l')} e^{-i \sum_1^r l_i \cdot \Theta k}$$

where ϕ is a real valued function. Thus,

$$\begin{aligned} \oint B |D|^{-n-2p+q} &= \text{Res}_{s=0} \sum_k' \sum_{l, l'} e^{i\phi(l, l')} \delta_{\sum_1^r l_i + l'_i, 0} \tilde{a}_l \tilde{b}_{l'} \frac{Q_\mu(k, l, l') e^{-i \sum_1^r l_i \cdot \Theta k}}{|k|^{s+2p+n-q}} \text{Tr}(\gamma^\mu) \\ &=: \text{Res}_{s=0} f_\mu(s) \text{Tr}(\gamma^\mu). \end{aligned}$$

We decompose $Q_\mu(k, l, l')$ as a sum $\sum_{h=0}^r M_{h, \mu}(l, l') Q_{h, \mu}(k)$ where $Q_{h, \mu}$ is a homogeneous polynomial in (k_1, \dots, k_n) and $M_{h, \mu}(l, l')$ is a polynomial in $((l_1)_1, \dots, (l_r)_n, (l'_1)_1, \dots, (l'_r)_n)$.

Similarly, we decompose $f_\mu(s)$ as $\sum_{h=0}^r f_{h, \mu}(s)$. Theorem 2.4.4 (ii) entails that $f_{h, \mu}(s)$ extends meromorphically to the whole complex plane \mathbb{C} with only one possible pole for $s+2p+n-q = n+d$ where $d := \deg Q_{h, \mu}$. In other words, if $d+q-2p \neq 0$, $f_{h, \mu}(s)$ is holomorphic at $s=0$. Suppose now $d+q-2p=0$ (note that this implies that d is odd, since q is odd by hypothesis), then, by Theorem 2.4.4 (ii)

$$\text{Res}_{s=0} f_{h, \mu}(s) = V \int_{u \in S^{n-1}} Q_{h, \mu}(u) dS(u)$$

where $V := \sum_{l, l' \in Z} M_{h, \mu}(l, l') e^{i\phi(l, l')} \delta_{\sum_1^r l_i + l'_i, 0} \tilde{a}_l \tilde{b}_{l'}$ and $Z := \{l, l' : \sum_{i=1}^r l_i = 0\}$. Since d is odd, $Q_{h, \mu}(-u) = -Q_{h, \mu}(u)$ and $\int_{u \in S^{n-1}} Q_{h, \mu}(u) dS(u) = 0$. Thus, $\text{Res}_{s=0} f_{h, \mu}(s) = 0$ in any case, which gives the result. \square

As we have seen, the crucial point of the preceding lemma is the decomposition of the numerator of the series $f_\mu(s)$ as polynomials in k . This has been possible because we restricted our pseudodifferential operators to $\Psi_1(\mathcal{A})$.

Proof of Proposition 2.2.6. The top element follows from Proposition 1.3.9 and according to (2.41),

$$\oint |D|^{-n} = \text{Res}_{s=0} \text{Tr}(|D|^{-s-n}) = 2^m \text{Res}_{s=0} Z_n(s+n) = \frac{2^{m+1} \pi^{n/2}}{\Gamma(n/2)}.$$

For the second equality, we get from Lemmas 2.2.8 and 1.3.6

$$\text{Res}_{s=n-k} \zeta_{DA}(s) = \sum_{p=1}^k \sum_{r_1, \dots, r_p=0}^{k-p} h(n-k, r, p) \oint \varepsilon^{r_1}(Y) \cdots \varepsilon^{r_p}(Y) |D|^{-(n-k)}.$$

Corollary 1.3.4 and Lemma 2.2.12 imply that $\oint \varepsilon^{r_1}(Y) \cdots \varepsilon^{r_p}(Y) |D|^{-(n-k)} = 0$, which gives the result.

Last equality follows from Lemma 2.2.11 and Corollary 1.3.11. \square

2.3 The spectral action

Here is the main result:

Theorem 2.3.1. *Consider the n -NC-torus $(C^\infty(\mathcal{T}_\Theta^n), \mathcal{H}, \mathcal{D})$ where $n \in \mathbb{N}$ and $\frac{1}{2\pi}\Theta$ is a $n \times n$ skew-symmetric real diophantine matrix, and a selfadjoint one-form $A = L(-iA_\alpha) \otimes \gamma^\alpha$. Then, the full spectral action of $\mathcal{D}_A = \mathcal{D} + A + \epsilon J A J^{-1}$ is*

(i) for $n = 2$,

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = 4\pi \Phi_2 \Lambda^2 + \mathcal{O}(\Lambda^{-2}),$$

(ii) for $n = 4$,

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = 8\pi^2 \Phi_4 \Lambda^4 - \frac{4\pi^2}{3} \Phi(0) \tau(F_{\mu\nu} F^{\mu\nu}) + \mathcal{O}(\Lambda^{-2}),$$

(iii) More generally, in

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = \sum_{k=0}^n \Phi_{n-k} c_{n-k}(A) \Lambda^{n-k} + \mathcal{O}(\Lambda^{-1}),$$

$c_{n-2}(A) = 0$, $c_{n-k}(A) = 0$ for k odd. In particular, $c_0(A) = 0$ when n is odd.

This result (for $n = 4$) has also been obtained in [62] using the heat kernel method. It is however interesting to get the result via direct computations of (1.6) since it shows how this formula is efficient. As we will see, the computation of all the noncommutative integrals require a lot of technical steps. One of the main points, namely to isolate where the Diophantine condition on Θ is assumed, is outlined here.

Remark 2.3.2. *Note that all terms must be gauge invariants, namely, according to (2.9), invariant by $A_\alpha \longrightarrow \gamma_u(A_\alpha) = u A_\alpha u^* + u \delta_\alpha(u^*)$. A particular case is $u = U_k$ where $U_k \delta_\alpha(U_k^*) = -ik_\alpha U_0$.*

In the same way, note that there is no contradiction with the commutative case where, for any selfadjoint one-form A , $\mathcal{D}_A = \mathcal{D}$ (so A is equivalent to 0!), since we assume in Theorem 2.3.1 that Θ is diophantine, so \mathcal{A} cannot be commutative.

Conjecture 2.3.3. *The constant term of the spectral action of \mathcal{D}_A on the noncommutative n -torus is proportional to the constant term of the spectral action of $\mathcal{D} + A$ on the commutative n -torus.*

Remark 2.3.4. *The appearance of a Diophantine condition for Θ has been characterized in dimension 2 by Connes [27, Prop. 49] where in this case, $\Theta = \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ with $\theta \in \mathbb{R}$. In fact, the Hochschild cohomology $H(\mathcal{A}_\Theta, \mathcal{A}_\Theta^*)$ satisfies $\dim H^j(\mathcal{A}_\Theta, \mathcal{A}_\Theta^*) = 2$ (or 1) for $j = 1$ (or $j = 2$) if and only if the irrational number θ satisfies a Diophantine condition like $|1 - e^{i2\pi n\theta}|^{-1} = \mathcal{O}(n^k)$ for some k .*

Recall that when the matrix Θ is quite irrational (see [68, Cor. 2.12]), then the C^ -algebra generated by \mathcal{A}_Θ is simple.*

Remark 2.3.5. *It is possible to generalize above theorem to the case $\mathcal{D} = -ig^\mu{}_\nu \delta_\mu \otimes \gamma^\nu$ instead of (2.4) when g is a positive definite constant matrix. The formulae in Theorem 2.3.1 are still valid.*

2.3.1 Computations of \oint

In order to get this theorem, let us prove a few technical lemmas.

We suppose from now on that Θ is a skew-symmetric matrix in $\mathcal{M}_n(\mathbb{R})$. No other hypothesis is assumed for Θ , except when it is explicitly stated.

When A is a selfadjoint one-form, we define for $n \in \mathbb{N}$, $q \in \mathbb{N}$, $2 \leq q \leq n$ and $\sigma \in \{-, +\}^q$

$$\begin{aligned}\mathbb{A}^+ &:= A\mathcal{D}D^{-2}, \\ \mathbb{A}^- &:= \epsilon JAJ^{-1}\mathcal{D}D^{-2}, \\ \mathbb{A}^\sigma &:= \mathbb{A}^{\sigma_q} \dots \mathbb{A}^{\sigma_1}.\end{aligned}$$

Lemma 2.3.6. *We have for any $q \in \mathbb{N}$,*

$$\oint (\tilde{A}D^{-1})^q = \oint (\tilde{A}\mathcal{D}D^{-2})^q = \sum_{\sigma \in \{+, -\}^q} \oint \mathbb{A}^\sigma.$$

Proof. Since $P_0 \in OP^{-\infty}$, $D^{-1} = \mathcal{D}D^{-2} \mod OP^{-\infty}$ and $\oint (\tilde{A}D^{-1})^q = \oint (\tilde{A}\mathcal{D}D^{-2})^q$. \square

Lemma 2.3.7. *Let A be a selfadjoint one-form, $n \in \mathbb{N}$ and $q \in \mathbb{N}$ with $2 \leq q \leq n$ and $\sigma \in \{-, +\}^q$. Then*

$$\oint \mathbb{A}^\sigma = \oint \mathbb{A}^{-\sigma}.$$

Proof. Let us first check that $JP_0 = P_0J$. Since $\mathcal{D}J = \epsilon J\mathcal{D}$, we get $\mathcal{D}JP_0 = 0$ so $JP_0 = P_0JP_0$. Since J is an antiunitary operator, we get $P_0J = P_0JP_0$ and finally, $P_0J = JP_0$. As a consequence, we get $J\mathcal{D}^2 = \mathcal{D}^2J$, $J\mathcal{D}D^{-2} = \epsilon\mathcal{D}D^{-2}J$, $J\mathbb{A}^+J^{-1} = \mathbb{A}^-$ and $J\mathbb{A}^-J^{-1} = \mathbb{A}^+$.

In summary, $J\mathbb{A}^{\sigma_i}J^{-1} = \mathbb{A}^{-\sigma_i}$.

The trace property of \oint now gives

$$\oint \mathbb{A}^\sigma = \oint \mathbb{A}^{\sigma_q} \dots \mathbb{A}^{\sigma_1} = \oint J\mathbb{A}^{\sigma_q}J^{-1} \dots J\mathbb{A}^{\sigma_1}J^{-1} \oint \mathbb{A}^{-\sigma_q} \dots \mathbb{A}^{-\sigma_1} = \oint \mathbb{A}^{-\sigma}. \quad \square$$

Definition 2.3.8. In [22] has been introduced the vanishing tadpole hypothesis:

$$\oint AD^{-1} = 0, \text{ for all } A \in \Omega_{\mathcal{D}}^1(\mathcal{A}). \quad (2.17)$$

By the following lemma, this condition is satisfied for the noncommutative torus, a fact more or less already known within the noncommutative community [137].

Lemma 2.3.9. *Let $n \in \mathbb{N}$, $A = L(-iA_\alpha) \otimes \gamma^\alpha = -i \sum_{l \in \mathbb{Z}^n} a_{\alpha,l} U_l \otimes \gamma^\alpha$, $A_\alpha \in \mathcal{A}_\Theta$, $\{a_{\alpha,l}\}_l \in \mathcal{S}(\mathbb{Z}^n)$, be a hermitian one-form. Then,*

(i) $\oint A^p D^{-q} = \oint (\epsilon JAJ^{-1})^p D^{-q} = 0$ for $p \geq 0$ and $1 \leq q < n$ (case $p = q = 1$ is tadpole hypothesis).

(ii) If $\frac{1}{2\pi}\Theta$ is diophantine, then $\oint BD^{-q} = 0$ for $1 \leq q < n$ and any B in the algebra generated by \mathcal{A} , $[\mathcal{D}, \mathcal{A}]$, JAJ^{-1} and $J[\mathcal{D}, \mathcal{A}]J^{-1}$.

Proof. (i) Let us compute

$$\oint A^p (\epsilon J A J^{-1})^{p'} D^{-q}.$$

With $A = L(-iA_\alpha) \otimes \gamma^\alpha$ and $\epsilon J A J^{-1} = R(iA_\alpha) \otimes \gamma^\alpha$, we get

$$A^p = L(-iA_{\alpha_1}) \cdots L(-iA_{\alpha_p}) \otimes \gamma^{\alpha_1} \cdots \gamma^{\alpha_p}$$

and

$$(\epsilon J A J^{-1})^{p'} = R(iA_{\alpha'_1}) \cdots R(iA_{\alpha'_{p'}}) \otimes \gamma^{\alpha'_1} \cdots \gamma^{\alpha'_{p'}}.$$

We note $\tilde{a}_{\alpha,l} := a_{\alpha_1,l_1} \cdots a_{\alpha_p,l_p}$. Since

$$L(-iA_{\alpha_1}) \cdots L(-iA_{\alpha_p}) R(iA_{\alpha'_1}) \cdots R(iA_{\alpha'_{p'}}) U_k = (-i)^p i^{p'} \sum_{l,l'} \tilde{a}_{\alpha,l} \tilde{a}_{\alpha',l'} U_{l_1} \cdots U_{l_p} U_k U_{l'_{p'}} \cdots U_{l'_1},$$

and

$$U_{l_1} \cdots U_{l_p} U_k = U_k U_{l_1} \cdots U_{l_p} e^{-i(\sum_i l_i) \cdot \Theta k},$$

we get, with

$$\begin{aligned} U_{l,l'} &:= U_{l_1} \cdots U_{l_p} U_{l'_{p'}} \cdots U_{l'_1}, \\ g_{\mu,\alpha,\alpha'}(s,k,l,l') &:= e^{ik \cdot \Theta \sum_j l_j} \frac{k_{\mu_1} \cdots k_{\mu_q}}{|k|^{s+2q}} \tilde{a}_{\alpha,l} \tilde{a}_{\alpha',l'}, \\ \gamma^{\alpha,\alpha',\mu} &:= \gamma^{\alpha_1} \cdots \gamma^{\alpha_p} \gamma^{\alpha'_1} \cdots \gamma^{\alpha'_{p'}} \gamma^{\mu_1} \cdots \gamma^{\mu_q}, \end{aligned}$$

$$A^p (\epsilon J A J^{-1})^{p'} D^{-q} |D|^{-s} U_k \otimes e_i \sim_c (-i)^p i^{p'} \sum_{l,l'} g_{\mu,\alpha,\alpha'}(s,k,l,l') U_k U_{l,l'} \otimes \gamma^{\alpha,\alpha',\mu} e_i.$$

Thus, $\oint A^p (\epsilon J A J^{-1})^{p'} D^{-q} = \text{Res}_{s=0} f(s)$ where

$$\begin{aligned} f(s) &:= \text{Tr} (A^p (\epsilon J A J^{-1})^{p'} D^{-q} |D|^{-s}) \\ &\sim_c (-i)^p i^{p'} \sum'_{k \in \mathbb{Z}^n} \langle U_k \otimes e_i, \sum_{l,l'} g_{\mu,\alpha,\alpha'}(s,k,l,l') U_k U_{l,l'} \otimes \gamma^{\alpha,\alpha',\mu} e_i \rangle \\ &\sim_c (-i)^p i^{p'} \sum'_{k \in \mathbb{Z}^n} \tau \left(\sum_{l,l'} g_{\mu,\alpha,\alpha'}(s,k,l,l') U_{l,l'} \right) \text{Tr}(\gamma^{\mu,\alpha,\alpha'}) \\ &\sim_c (-i)^p i^{p'} \sum'_{k \in \mathbb{Z}^n} \sum_{l,l'} g_{\mu,\alpha,\alpha'}(s,k,l,l') \tau(U_{l,l'}) \text{Tr}(\gamma^{\mu,\alpha,\alpha'}). \end{aligned}$$

It is straightforward to check that the series $\sum'_{k,l,l'} g_{\mu,\alpha,\alpha'}(s,k,l,l') \tau(U_{l,l'})$ is absolutely summable if $\Re(s) > R$ for a $R > 0$. Thus, we can exchange the summation on k and l, l' , which gives

$$f(s) \sim_c (-i)^p i^{p'} \sum_{l,l'} \sum'_{k \in \mathbb{Z}^n} g_{\mu,\alpha,\alpha'}(s,k,l,l') \tau(U_{l,l'}) \text{Tr}(\gamma^{\mu,\alpha,\alpha'}).$$

If we suppose now that $p' = 0$, we see that,

$$f(s) \sim_c (-i)^p \sum_l \sum'_{k \in \mathbb{Z}^n} \frac{k_{\mu_1} \cdots k_{\mu_q}}{|k|^{s+2q}} \tilde{a}_{\alpha,l} \delta_{\sum_{i=1}^p l_i, 0} \text{Tr}(\gamma^{\mu,\alpha,\alpha'})$$

which is, by Proposition 2.4.15, analytic at 0. In particular, for $p = q = 1$, we see that $\oint AD^{-1} = 0$, i.e. the vanishing tadpole hypothesis is satisfied. Similarly, if we suppose $p = 0$, we get

$$f(s) \sim_c (-i)^{p'} \sum_{l'} \sum_{k \in \mathbb{Z}^n} \frac{k_{\mu_1} \dots k_{\mu_q}}{|k|^{s+2q}} \tilde{a}_{\alpha, l'} \delta_{\sum_{i=1}^{p'} l'_i, 0} \text{Tr}(\gamma^{\mu, \alpha, \alpha'})$$

which is holomorphic at 0.

(ii) Adapting the proof of Lemma 2.2.12 to our setting (taking $q_i = 0$, and adding gamma matrices components), we see that

$$\oint B D^{-q} = \text{Res}_{s=0} \sum_{k'} \sum_{l, l'} e^{i\phi(l, l')} \delta_{\sum_{i=1}^r l_i + l'_i, 0} \tilde{a}_{\alpha, l} \tilde{b}_{\beta, l'} \frac{k_{\mu_1} \dots k_{\mu_q} e^{-i \sum_{i=1}^r l_i \cdot \Theta k}}{|k|^{s+2q}} \text{Tr}(\gamma^{(\mu, \alpha, \beta)})$$

where $\gamma^{(\mu, \alpha, \beta)}$ is a complicated product of gamma matrices. By Theorem 2.4.4 (ii), since we suppose here that $\frac{1}{2\pi}\Theta$ is diophantine, this residue is 0. \square

Even dimensional case

Corollary 2.3.10. *Same hypothesis as in Lemma 2.3.9.*

(i) Case $n = 2$:

$$\oint A^q D^{-q} = -\delta_{q,2} 4\pi \tau(A_\alpha A^\alpha).$$

(ii) Case $n = 4$: with the shorthand $\delta_{\mu_1, \dots, \mu_4} := \delta_{\mu_1 \mu_2} \delta_{\mu_3 \mu_4} + \delta_{\mu_1 \mu_3} \delta_{\mu_2 \mu_4} + \delta_{\mu_1 \mu_4} \delta_{\mu_2 \mu_3}$,

$$\oint A^q D^{-q} = \delta_{q,4} \frac{\pi^2}{12} \tau(A_{\alpha_1} \dots A_{\alpha_4}) \text{Tr}(\gamma^{\alpha_1} \dots \gamma^{\alpha_4} \gamma^{\mu_1} \dots \gamma^{\mu_4}) \delta_{\mu_1, \dots, \mu_4}.$$

Proof. (i, ii) The same computation as in Lemma 2.3.9 (i) (with $p' = 0$, $p = q = n$) gives

$$\oint A^n D^{-n} = \text{Res}_{s=0} (-i)^n \left(\sum_{k \in \mathbb{Z}^n} \frac{k_{\mu_1} \dots k_{\mu_n}}{|k|^{s+2n}} \right) \tau \left(\sum_{l \in (\mathbb{Z}^n)^n} \tilde{a}_{\alpha, l} U_{l_1} \dots U_{l_n} \right) \text{Tr}(\gamma^{\alpha_1} \dots \gamma^{\alpha_n} \gamma^{\mu_1} \dots \gamma^{\mu_n})$$

and the result follows from Proposition 2.4.15. \square

We will use few notations:

If $n \in \mathbb{N}$, $q \geq 2$, $l := (l_1, \dots, l_{q-1}) \in (\mathbb{Z}^n)^{q-1}$, $\alpha := (\alpha_1, \dots, \alpha_q) \in \{1, \dots, n\}^q$, $k \in \mathbb{Z}^n \setminus \{0\}$, $\sigma \in \{-, +\}^q$, $(a_i)_{1 \leq i \leq n} \in (\mathcal{S}(\mathbb{Z}^n))^n$,

$$\begin{aligned} l_q &:= - \sum_{1 \leq j \leq q-1} l_j, \quad \lambda_\sigma := (-i)^q \prod_{j=1 \dots q} \sigma_j, \quad \tilde{a}_{\alpha, l} := a_{\alpha_1, l_1} \dots a_{\alpha_q, l_q}, \\ \phi_\sigma(k, l) &:= \sum_{1 \leq j \leq q-1} (\sigma_j - \sigma_q) k \cdot \Theta l_j + \sum_{2 \leq j \leq q-1} \sigma_j (l_1 + \dots + l_{j-1}) \cdot \Theta l_j, \\ g_\mu(s, k, l) &:= \frac{k_{\mu_1} (k+l_1)_{\mu_2} \dots (k+l_1+\dots+l_{q-1})_{\mu_q}}{|k|^{s+2} |k+l_1|^2 \dots |k+l_1+\dots+l_{q-1}|^2}, \end{aligned}$$

with the convention $\sum_{2 \leq j \leq q-1} = 0$ when $q = 2$, and $g_\mu(s, k, l) = 0$ whenever $\widehat{l}_i = -k$ for a $1 \leq i \leq q-1$.

Lemma 2.3.11. *Let $A = L(-iA_\alpha) \otimes \gamma^\alpha = -i \sum_{l \in \mathbb{Z}^n} a_{\alpha,l} U_l \otimes \gamma^\alpha$ where $A_\alpha = -A_\alpha^* \in \mathcal{A}_\Theta$ and $\{a_{\alpha,l}\}_l \in \mathcal{S}(\mathbb{Z}^n)$, with $n \in \mathbb{N}$, be a hermitian one-form, and let $2 \leq q \leq n$, $\sigma \in \{-, +\}^q$.*

Then, $\oint \mathbb{A}^\sigma = \text{Res}_{s=0} f(s)$ where

$$f(s) := \sum_{l \in (\mathbb{Z}^n)^{q-1}} \sum'_{k \in \mathbb{Z}^n} \lambda_\sigma e^{\frac{i}{2}\phi_\sigma(k,l)} g_\mu(s, k, l) \tilde{a}_{\alpha,l} \text{Tr}(\gamma^{\alpha_q} \gamma^{\mu_q} \dots \gamma^{\alpha_1} \gamma^{\mu_1}).$$

Proof. By definition, $\oint \mathbb{A}^\sigma = \text{Res}_{s=0} f(s)$ where

$$\text{Tr}(\mathbb{A}^{\sigma_q} \dots \mathbb{A}^{\sigma_1} |D|^{-s}) \sim_c \sum'_{k \in \mathbb{Z}^n} \langle U_k \otimes e^i, |k|^{-s} \mathbb{A}^{\sigma_q} \dots \mathbb{A}^{\sigma_1} U_k \otimes e_i \rangle =: f(s).$$

Let $r \in \mathbb{Z}^n$ and $v \in \mathbb{C}^{2^m}$. Since $A = L(-iA_\alpha) \otimes \gamma^\alpha$, and $\epsilon J A J^{-1} = R(iA_\alpha) \otimes \gamma^\alpha$, we get

$$\begin{aligned} \mathbb{A}^+ U_r \otimes v &= A D D^{-2} U_r \otimes v = A \frac{r_\mu}{|r|^2 + \delta_{r,0}} U_r \otimes \gamma^\mu v = -i \frac{r_\mu}{|r|^2 + \delta_{r,0}} A_\alpha U_r \otimes \gamma^\alpha \gamma^\mu v, \\ \mathbb{A}^- U_r \otimes v &= \epsilon J A J^{-1} D D^{-2} U_r \otimes v = \epsilon J A J^{-1} \frac{r_\mu}{|r|^2 + \delta_{r,0}} U_r \otimes \gamma^\mu v = i \frac{r_\mu}{|r|^2 + \delta_{r,0}} U_r A_\alpha \otimes \gamma^\alpha \gamma^\mu v. \end{aligned}$$

With $U_l U_r = e^{\frac{i}{2}r \cdot \Theta l} U_{r+l}$ and $U_r U_l = e^{-\frac{i}{2}r \cdot \Theta l} U_{r+l}$, we obtain, for any $1 \leq j \leq q$,

$$\mathbb{A}^{\sigma_j} U_r \otimes v = \sum_{l \in \mathbb{Z}^n} (-\sigma_j) i e^{\sigma_j \frac{i}{2}r \cdot \Theta l} \frac{r_\mu}{|r|^2 + \delta_{r,0}} a_{\alpha,l} U_{r+l} \otimes \gamma^\alpha \gamma^\mu v.$$

We now apply q times this formula to get

$$|k|^{-s} \mathbb{A}^{\sigma_q} \dots \mathbb{A}^{\sigma_1} U_k \otimes e_i = \lambda_\sigma \sum_{l \in (\mathbb{Z}^n)^q} e^{\frac{i}{2}\phi_\sigma(k,l)} g_\mu(s, k, l) \tilde{a}_{\alpha,l} U_{k+\sum_j l_j} \otimes \gamma^{\alpha_q} \gamma^{\mu_q} \dots \gamma^{\alpha_1} \gamma^{\mu_1} e_i$$

with

$$\phi_\sigma(k, l) := \sigma_1 k \cdot \Theta l_1 + \sigma_2 (k + l_1) \cdot \Theta l_2 + \dots + \sigma_q (k + l_1 + \dots + l_{q-1}) \cdot \Theta l_q.$$

Thus,

$$\begin{aligned} f(s) &= \sum'_{k \in \mathbb{Z}^n} \tau(\lambda_\sigma \sum_{l \in (\mathbb{Z}^n)^q} e^{\frac{i}{2}\phi_\sigma(k,l)} g_\mu(s, k, l) \tilde{a}_{\alpha,l} U_{\sum_j l_j} e^{\frac{i}{2}k \cdot \Theta \sum_j l_j}) \text{Tr}(\gamma^{\alpha_q} \gamma^{\mu_q} \dots \gamma^{\alpha_1} \gamma^{\mu_1}) \\ &= \sum'_{k \in \mathbb{Z}^n} \lambda_\sigma \sum_{l \in (\mathbb{Z}^n)^q} e^{\frac{i}{2}\phi_\sigma(k,l)} g_\mu(s, k, l) \tilde{a}_{\alpha,l} \delta(\sum_j l_j) \text{Tr}(\gamma^{\alpha_q} \gamma^{\mu_q} \dots \gamma^{\alpha_1} \gamma^{\mu_1}) \\ &= \sum'_{k \in \mathbb{Z}^n} \lambda_\sigma \sum_{l \in (\mathbb{Z}^n)^{q-1}} e^{\frac{i}{2}\phi_\sigma(k,l)} g_\mu(s, k, l) \tilde{a}_{\alpha,l} \text{Tr}(\gamma^{\alpha_q} \gamma^{\mu_q} \dots \gamma^{\alpha_1} \gamma^{\mu_1}) \end{aligned}$$

where in the last sum l_q is fixed to $-\sum_{1 \leq j \leq q-1} l_j$ and thus,

$$\phi_\sigma(k, l) = \sum_{1 \leq j \leq q-1} (\sigma_j - \sigma_q) k \cdot \Theta l_j + \sum_{2 \leq j \leq q-1} \sigma_j (l_1 + \dots + l_{j-1}) \cdot \Theta l_j.$$

By Lemma 2.4.9, there exists a $R > 0$ such that for any $s \in \mathbb{C}$ with $\Re(s) > R$, the family

$$(e^{\frac{i}{2}\phi_\sigma(k,l)} g_\mu(s, k, l) \tilde{a}_{\alpha,l})_{(k,l) \in (\mathbb{Z}^n \setminus \{0\}) \times (\mathbb{Z}^n)^{q-1}}$$

is absolutely summable as a linear combination of families of the type considered in that lemma. As a consequence, we can exchange the summations on k and l , which gives the result. \square

In the following, we will use the shorthand

$$c := \frac{4\pi^2}{3}.$$

Lemma 2.3.12. *Suppose $n = 4$. Then, with the same hypothesis of Lemma 2.3.11,*

$$\begin{aligned} (i) \quad & \frac{1}{2} \oint (\mathbb{A}^+)^2 = \frac{1}{2} \oint (\mathbb{A}^-)^2 = c \sum_{l \in \mathbb{Z}^4} a_{\alpha_1, l} a_{\alpha_2, -l} (l^{\alpha_1} l^{\alpha_2} - \delta^{\alpha_1 \alpha_2} |l|^2). \\ (ii) \quad & -\frac{1}{3} \oint (\mathbb{A}^+)^3 = -\frac{1}{3} \oint (\mathbb{A}^-)^3 = 4c \sum_{l_i \in \mathbb{Z}^4} a_{\alpha_3, -l_1 - l_2} a_{l_2}^{\alpha_1} a_{\alpha_1, l_1} \sin \frac{l_1 \cdot \Theta l_2}{2} l_1^{\alpha_3}. \\ (iii) \quad & \frac{1}{4} \oint (\mathbb{A}^+)^4 = \frac{1}{4} \oint (\mathbb{A}^-)^4 = 2c \sum_{l_i \in \mathbb{Z}^4} a_{\alpha_1, -l_1 - l_2 - l_3} a_{\alpha_2, l_3} a_{l_2}^{\alpha_1} a_{l_1}^{\alpha_2} \sin \frac{l_1 \cdot \Theta (l_2 + l_3)}{2} \sin \frac{l_2 \cdot \Theta l_3}{2}. \end{aligned}$$

(iv) Suppose $\frac{1}{2\pi}\Theta$ diophantine. Then the crossed terms in $\oint (\mathbb{A}^+ + \mathbb{A}^-)^q$ vanish: if C is the set of all $\sigma \in \{-, +\}^q$ with $2 \leq q \leq 4$, such that there exist i, j satisfying $\sigma_i \neq \sigma_j$, we have $\sum_{\sigma \in C} \oint \mathbb{A}^\sigma = 0$.

Proof. (i) Lemma 2.3.11 entails that $\oint \mathbb{A}^{++} = \text{Res}_{s=0} \sum_{l \in \mathbb{Z}^n} -f(s, l)$ where

$$f(s, l) := \sum'_{k \in \mathbb{Z}^n} \frac{k_{\mu_1}(k+l)_{\mu_2}}{|k|^{s+2}|k+l|^2} \tilde{a}_{\alpha, l} \text{Tr}(\gamma^{\alpha_2} \gamma^{\mu_2} \gamma^{\alpha_1} \gamma^{\mu_1}) \text{ and } \tilde{a}_{\alpha, l} := a_{\alpha_1, l} a_{\alpha_2, -l}.$$

We will now reduce the computation of the residue of an expression involving terms like $|k+l|^2$ in the denominator to the computation of residues of zeta functions. To proceed, we use (2.37) into an expression like the one appearing in $f(s, l)$. We see that the last term on the righthandside yields a $Z_n(s)$ while the first one is less divergent by one power of k . If this is not enough, we repeat this operation for the new factor of $|k+l|^2$ in the denominator. For $f(s, l)$, which is quadratically divergent at $s = 0$, we have to repeat this operation three times before ending with a convergent result. All the remaining terms are expressible in terms of Z_n functions. We get, using three times (2.37),

$$\frac{1}{|k+l|^2} = \frac{1}{|k|^2} - \frac{2k \cdot l + |l|^2}{|k|^4} + \frac{(2k \cdot l + |l|^2)^2}{|k|^6} - \frac{(2k \cdot l + |l|^2)^3}{|k|^6 |k+l|^2}. \quad (2.18)$$

Let us define

$$f_{\alpha, \mu}(s, l) := \sum'_{k \in \mathbb{Z}^n} \frac{k_{\mu_1}(k+l)_{\mu_2}}{|k|^{s+2}|k+l|^2} \tilde{a}_{\alpha, l}$$

so that $f(s, l) = f_{\alpha, \mu}(s, l) \text{Tr}(\gamma^{\alpha_2} \gamma^{\mu_2} \gamma^{\alpha_1} \gamma^{\mu_1})$. Equation (2.18) gives

$$f_{\alpha, \mu}(s, l) = f_1(s, l) - f_2(s, l) + f_3(s, l) - r(s, l)$$

with obvious identifications. Note that the function

$$r(s, l) = \sum'_{k \in \mathbb{Z}^n} \frac{k_{\mu_1}(k+l)_{\mu_2} (2kl + |l|^2)^3}{|k|^{s+8}|k+l|^2} \tilde{a}_{\alpha, l}$$

is a linear combination of functions of the type $H(s, l)$ satisfying the hypothesis of Corollary 2.4.12. Thus, $r(s, l)$ satisfies (H1) and with the previously seen equivalence relation modulo functions satisfying this hypothesis we get $f_{\alpha, \mu}(s, l) \sim f_1(s, l) - f_2(s, l) + f_3(s, l)$.

Let's now compute $f_1(s, l)$.

$$f_1(s, l) = \sum_{k \in \mathbb{Z}^n} ' \frac{k_{\mu_1}(k+l)_{\mu_2}}{|k|^{s+4}} \tilde{a}_{\alpha, l} = \tilde{a}_{\alpha, l} \sum_{k \in \mathbb{Z}^n} ' \frac{k_{\mu_1} k_{\mu_2}}{|k|^{s+4}} + 0.$$

Proposition 2.4.1 entails that $s \mapsto \sum_{k \in \mathbb{Z}^n} ' \frac{k_{\mu_1} k_{\mu_2}}{|k|^{s+4}}$ is holomorphic at 0. Thus, $f_1(s, l)$ satisfies (H1), and $f_{\alpha, \mu}(s, l) \sim -f_2(s, l) + f_3(s, l)$.

Let's now compute $f_2(s, l)$ modulo (H1). We get, using several times Proposition 2.4.1,

$$\begin{aligned} f_2(s, l) &= \sum_{k \in \mathbb{Z}^n} ' \frac{k_{\mu_1}(k+l)_{\mu_2}(2kl+|l|^2)}{|k|^{s+6}} \tilde{a}_{\alpha, l} = \sum_{k \in \mathbb{Z}^n} ' \frac{(2kl)k_{\mu_1}k_{\mu_2} + (2kl)k_{\mu_1}l_{\mu_2} + |l|^2k_{\mu_1}k_{\mu_2} + l_{\mu_2}|l|^2k_{\mu_1}}{|k|^{s+6}} \tilde{a}_{\alpha, l} \\ &\sim 0 + \sum_{k \in \mathbb{Z}^n} ' \frac{(2kl)k_{\mu_1}l_{\mu_2}}{|k|^{s+6}} \tilde{a}_{\alpha, l} + \sum_{k \in \mathbb{Z}^n} ' \frac{|l|^2k_{\mu_1}k_{\mu_2}}{|k|^{s+6}} \tilde{a}_{\alpha, l} + 0. \end{aligned}$$

Recall that $\sum_{k \in \mathbb{Z}^n} ' \frac{k_i k_j}{|k|^{s+6}} = \frac{\delta_{ij}}{n} Z_n(s+4)$. Thus,

$$f_2(s, l) \sim 2l^i l_{\mu_2} \tilde{a}_{\alpha, l} \frac{\delta_{i\mu_1}}{n} Z_n(s+4) + |l|^2 \tilde{a}_{\alpha, l} \frac{\delta_{\mu_1\mu_2}}{n} Z_n(s+4).$$

Finally, let us compute $f_3(s, l)$ modulo (H1) following the same principles:

$$\begin{aligned} f_3(s, l) &= \sum_{k \in \mathbb{Z}^n} ' \frac{k_{\mu_1}(k+l)_{\mu_2}(2kl+|l|^2)^2}{|k|^{s+8}} \tilde{a}_{\alpha, l} \\ &= \sum_{k \in \mathbb{Z}^n} ' \frac{(2kl)^2 k_{\mu_1} k_{\mu_2} + (2kl)^2 k_{\mu_1} l_{\mu_2} + |l|^4 k_{\mu_1} k_{\mu_2} + |l|^4 k_{\mu_1} l_{\mu_2} + (4kl)|l|^2 k_{\mu_1} k_{\mu_2} + (4kl)|l|^2 k_{\mu_1} l_{\mu_2}}{|k|^{s+8}} \tilde{a}_{\alpha, l} \\ &\sim 4l^i l^j \sum_{k \in \mathbb{Z}^n} ' \frac{k_i k_j k_{\mu_1} k_{\mu_2}}{|k|^{s+8}} \tilde{a}_{\alpha, l} + 0. \end{aligned}$$

In conclusion,

$$f_{\alpha, \mu}(s, l) \sim -\frac{1}{4}(2l_{\mu_1} l_{\mu_2} + |l|^2 \delta_{\mu_1\mu_2}) \tilde{a}_{\alpha, l} Z_n(s+4) + 4l^i l^j \tilde{a}_{\alpha, l} \sum_{k \in \mathbb{Z}^n} ' \frac{k_i k_j k_{\mu_1} k_{\mu_2}}{|k|^{s+8}} =: g_{\alpha, \mu}(s, l).$$

Proposition (2.4.1) entails that $Z_n(s+4)$ and $s \mapsto \sum_{k \in \mathbb{Z}^n} ' \frac{k_i k_j k_{\mu_1} k_{\mu_2}}{|k|^{s+8}}$ extend holomorphically in a punctured open disk centered at 0. Thus, $g_{\alpha, \mu}(s, l)$ satisfies (H2) and we can apply Lemma 2.4.13 to get

$$-\oint (\mathbb{A}^+)^2 = \text{Res}_{s=0} \sum_{l \in \mathbb{Z}^n} f(s, l) = \sum_{l \in \mathbb{Z}^n} \text{Res}_{s=0} g_{\alpha, \mu}(s, l) \text{Tr}(\gamma^{\alpha_2} \gamma^{\mu_2} \gamma^{\alpha_1} \gamma^{\mu_1}) =: \sum_{l \in \mathbb{Z}^n} \text{Res}_{s=0} g(s, l).$$

The problem is now reduced to the computation of $\text{Res}_{s=0} g(s, l)$. Recall that $\text{Res}_{s=0} Z_4(s+4) = 2\pi^2$ by (2.41) or (2.38), and

$$\text{Res}_{s=0} \sum_{k \in \mathbb{Z}^n} ' \frac{k_i k_j k_l k_m}{|k|^{s+8}} = (\delta_{ij} \delta_{lm} + \delta_{il} \delta_{jm} + \delta_{im} \delta_{jl}) \frac{\pi^2}{12}.$$

Thus,

$$\text{Res}_{s=0} g_{\alpha, \mu}(s, l) = -\frac{\pi^2}{3} \tilde{a}_{\alpha, l} (l_{\mu_1} l_{\mu_2} + \frac{1}{2} |l|^2 \delta_{\mu_1\mu_2}).$$

We will use

$$\mathrm{Tr}(\gamma^{\mu_1} \dots \gamma^{\mu_{2j}}) = \mathrm{Tr}(1) \sum_{\text{all pairings of } \{1 \dots 2j\}} s(P) \delta_{\mu_{P_1} \mu_{P_2}} \delta_{\mu_{P_3} \mu_{P_4}} \dots \delta_{\mu_{P_{2j-1}} \mu_{P_{2j}}} \quad (2.19)$$

where $s(P)$ is the signature of the permutation P when $P_{2m-1} < P_{2m}$ for $1 \leq m \leq n$. This gives

$$\mathrm{Tr}(\gamma^{\alpha_2} \gamma^{\mu_2} \gamma^{\alpha_1} \gamma^{\mu_1}) = 2^m (\delta^{\alpha_2 \mu_2} \delta^{\alpha_1 \mu_1} - \delta^{\alpha_1 \alpha_2} \delta^{\mu_2 \mu_1} + \delta^{\alpha_2 \mu_1} \delta^{\mu_2 \alpha_1}). \quad (2.20)$$

Thus,

$$\begin{aligned} \mathrm{Res}_{s=0} g(s, l) &= -c \tilde{a}_{\alpha, l} (l_{\mu_1} l_{\mu_2} + \frac{1}{2} |l|^2 \delta_{\mu_1 \mu_2}) (\delta^{\alpha_2 \mu_2} \delta^{\alpha_1 \mu_1} - \delta^{\alpha_1 \alpha_2} \delta^{\mu_2 \mu_1} + \delta^{\alpha_2 \mu_1} \delta^{\mu_2 \alpha_1}) \\ &= -2c \tilde{a}_{\alpha, l} (l^{\alpha_1} l^{\alpha_2} - \delta^{\alpha_1 \alpha_2} |l|^2). \end{aligned}$$

Finally,

$$\frac{1}{2} \oint (\mathbb{A}^+)^2 = \frac{1}{2} \oint (\mathbb{A}^-)^2 = c \sum_{l \in \mathbb{Z}^n} a_{\alpha_1, l} a_{\alpha_2, -l} (l^{\alpha_1} l^{\alpha_2} - \delta^{\alpha_1 \alpha_2} |l|^2).$$

(ii) Lemma 2.3.11 entails that $\oint \mathbb{A}^{+++} = \mathrm{Res}_{s=0} \sum_{(l_1, l_2) \in (\mathbb{Z}^n)^2} f(s, l)$ where

$$\begin{aligned} f(s, l) &:= \sum'_{k \in \mathbb{Z}^n} i e^{\frac{i}{2} l_1 \Theta l_2} \frac{k_{\mu_1} (k+l_1)_{\mu_2} (k+\widehat{l}_2)_{\mu_3}}{|k|^{s+2} |k+l_1|^2 |k+\widehat{l}_2|^2} \tilde{a}_{\alpha, l} \mathrm{Tr}(\gamma^{\alpha_3} \gamma^{\mu_3} \gamma^{\alpha_2} \gamma^{\mu_2} \gamma^{\alpha_1} \gamma^{\mu_1}) \\ &=: f_{\alpha, \mu}(s, l) \mathrm{Tr}(\gamma^{\alpha_3} \gamma^{\mu_3} \gamma^{\alpha_2} \gamma^{\mu_2} \gamma^{\alpha_1} \gamma^{\mu_1}), \end{aligned}$$

and $\tilde{a}_{\alpha, l} := a_{\alpha_1, l_1} a_{\alpha_2, l_2} a_{\alpha_3, -\widehat{l}_2}$ with $\widehat{l}_2 := l_1 + l_2$.

We use the same technique as in (i):

$$\begin{aligned} \frac{1}{|k+l_1|^2} &= \frac{1}{|k|^2} - \frac{2k \cdot l_1 + |l_1|^2}{|k|^4} + \frac{(2k \cdot l_1 + |l_1|^2)^2}{|k|^4 |k+l_1|^2}, \\ \frac{1}{|k+\widehat{l}_2|^2} &= \frac{1}{|k|^2} - \frac{2k \cdot \widehat{l}_2 + |\widehat{l}_2|^2}{|k|^4} + \frac{(2k \cdot \widehat{l}_2 + |\widehat{l}_2|^2)^2}{|k|^4 |k+\widehat{l}_2|^2} \end{aligned}$$

and thus,

$$\frac{1}{|k+l_1|^2 |k+\widehat{l}_2|^2} = \frac{1}{|k|^4} - \frac{2k \cdot l_1}{|k|^6} - \frac{2k \cdot \widehat{l}_2}{|k|^6} + R(k, l) \quad (2.21)$$

where the remain $R(k, l)$ is a term of order at most -6 in k . Equation (2.21) gives

$$f_{\alpha, \mu}(s, l) = f_1(s, l) + r(s, l)$$

where $r(s, l)$ corresponds to $R(k, l)$. Note that the function

$$r(s, l) = \sum'_{k \in \mathbb{Z}^n} i e^{\frac{i}{2} l_1 \Theta l_2} \frac{k_{\mu_1} (k+l_1)_{\mu_2} (k+\widehat{l}_2)_{\mu_3} R(k, l)}{|k|^{s+2}} \tilde{a}_{\alpha, l}$$

is a linear combination of functions of the type $H(s, l)$ satisfying the hypothesis of Corollary (2.4.12). Thus, $r(s, l)$ satisfies (H1) and $f_{\alpha, \mu}(s, l) \sim f_1(s, l)$.

Let us compute $f_1(s, l)$ modulo (H1)

$$\begin{aligned}
f_1(s, l) &= \sum'_{k \in \mathbb{Z}^n} i e^{\frac{i}{2} l_1 \Theta l_2} \frac{k_{\mu_1} (k+l_1)_{\mu_2} (k+\widehat{l}_2)_{\mu_3}}{|k|^{s+6}} \tilde{a}_{\alpha, l} - \sum'_{k \in \mathbb{Z}^n} i e^{\frac{i}{2} l_1 \Theta l_2} \frac{k_{\mu_1} (k+l_1)_{\mu_2} (k+\widehat{l}_2)_{\mu_3} (2k.l_1 + 2k.\widehat{l}_2)}{|k|^{s+8}} \tilde{a}_{\alpha, l} \\
&\sim \sum'_{k \in \mathbb{Z}^n} i e^{\frac{i}{2} l_1 \Theta l_2} \frac{k_{\mu_1} k_{\mu_2} \widehat{l}_{2\mu_3} + k_{\mu_1} k_{\mu_3} l_{1\mu_2}}{|k|^{s+6}} \tilde{a}_{\alpha, l} - \sum'_{k \in \mathbb{Z}^n} i e^{\frac{i}{2} l_1 \Theta l_2} \frac{k_{\mu_1} k_{\mu_2} k_{\mu_3} (2k.l_1 + 2k.\widehat{l}_2)}{|k|^{s+8}} \tilde{a}_{\alpha, l} \\
&= i e^{\frac{i}{2} l_1 \Theta l_2} \tilde{a}_{\alpha, l} ((l_{1\mu_2} \delta_{\mu_1\mu_3} + \widehat{l}_{2\mu_3} \delta_{\mu_1\mu_2}) \frac{1}{4} Z_4(s+4) - 2(l_1^i + \widehat{l}_2^i) \sum'_{k \in \mathbb{Z}^n} \frac{k_{\mu_1} k_{\mu_2} k_{\mu_3} k_i}{|k|^{s+8}}) \\
&=: g_{\alpha, \mu}(s, l).
\end{aligned}$$

Since $g_{\alpha, \mu}(s, l)$ satisfies (H2), we can apply Lemma 2.4.13 to get

$$\begin{aligned}
\oint (\mathbb{A}^+)^3 &= \text{Res}_{s=0} \sum_{(l_1, l_2) \in (\mathbb{Z}^n)^2} f(s, l) \\
&= \sum_{(l_1, l_2) \in (\mathbb{Z}^n)^2} \text{Res}_{s=0} g_{\alpha, \mu}(s, l) \text{Tr}(\gamma^{\alpha_3} \gamma^{\mu_3} \gamma^{\alpha_2} \gamma^{\mu_2} \gamma^{\alpha_1} \gamma^{\mu_1}) =: \sum_l X_l.
\end{aligned}$$

Recall that $l_3 := -l_1 - l_2 = -\widehat{l}_2$. By (2.38) and (2.40),

$$\begin{aligned}
\text{Res}_{s=0} g_{\alpha, \mu}(s, l) i e^{\frac{i}{2} l_1 \Theta l_2} \tilde{a}_{\alpha, l} &(2(-l_1^i + l_3^i) \frac{\pi^2}{12} (\delta_{\mu_1\mu_2} \delta_{\mu_3 i} + \delta_{\mu_1\mu_3} \delta_{\mu_2 i} + \delta_{\mu_1 i} \delta_{\mu_2\mu_3}) \\
&+ (l_{1\mu_2} \delta_{\mu_1\mu_3} - l_{3\mu_3} \delta_{\mu_1\mu_2}) \frac{\pi^2}{2}).
\end{aligned}$$

We decompose X_l in five terms: $X_l = 2^m \frac{\pi^2}{2} i e^{\frac{i}{2} l_1 \Theta l_2} \tilde{a}_{\alpha, l} (T_1 + T_2 + T_3 + T_4 + T_5)$ where

$$\begin{aligned}
T_0 &:= \frac{1}{3}(-l_1^i + l_3^i) (\delta_{\mu\nu} \delta_{\rho i} + \delta_{\mu\rho} \delta_{\nu i} + \delta_{\mu i} \delta_{\nu\rho}) + l_{1\nu} \delta_{\mu\rho} - l_{3\rho} \delta_{\mu\nu}, \\
T_1 &:= (\delta^{\alpha_3\rho} \delta^{\alpha_2\nu} \delta^{\alpha_1\mu} - \delta^{\alpha_3\rho} \delta^{\alpha_2\alpha_1} \delta^{\mu\nu} + \delta^{\alpha_3\rho} \delta^{\alpha_2\mu} \delta^{\alpha_1\nu}) T_0, \\
T_2 &:= (-\delta^{\alpha_2\alpha_3} \delta^{\rho\nu} \delta^{\alpha_1\mu} + \delta^{\alpha_2\alpha_3} \delta^{\alpha_1\rho} \delta^{\mu\nu} - \delta^{\alpha_2\alpha_3} \delta^{\rho\mu} \delta^{\alpha_1\nu}) T_0, \\
T_3 &:= (\delta^{\alpha_3\nu} \delta^{\alpha_2\rho} \delta^{\alpha_1\mu} - \delta^{\alpha_3\nu} \delta^{\alpha_1\rho} \delta^{\alpha_2\mu} + \delta^{\alpha_3\nu} \delta^{\rho\mu} \delta^{\alpha_1\alpha_2}) T_0, \\
T_4 &:= (-\delta^{\alpha_1\alpha_3} \delta^{\alpha_2\rho} \delta^{\mu\nu} + \delta^{\alpha_1\alpha_3} \delta^{\rho\nu} \delta^{\alpha_2\mu} - \delta^{\alpha_1\alpha_3} \delta^{\rho\mu} \delta^{\alpha_2\nu}) T_0, \\
T_5 &:= (\delta^{\alpha_3\mu} \delta^{\alpha_2\rho} \delta^{\alpha_1\nu} - \delta^{\alpha_3\mu} \delta^{\rho\nu} \delta^{\alpha_1\alpha_2} + \delta^{\alpha_3\mu} \delta^{\alpha_1\rho} \delta^{\alpha_2\nu}) T_0.
\end{aligned}$$

With the shorthand $p := -l_1 - 2l_3$, $q := 2l_1 + l_3$, $r := -p - q = -l_1 + l_3$, we compute each T_i , and find

$$\begin{aligned}
3T_1 &= \delta^{\alpha_1\alpha_2} (2 - 2^m) p^{\alpha_3} + \delta^{\alpha_3\alpha_1} q^{\alpha_2} - \delta^{\alpha_2\alpha_1} q^{\alpha_3} + \delta^{\alpha_3\alpha_2} q^{\alpha_1} + \delta^{\alpha_3\alpha_2} r^{\alpha_1} - \delta^{\alpha_2\alpha_1} r^{\alpha_3} + \delta^{\alpha_3\alpha_1} r^{\alpha_2}, \\
3T_2 &= (2^m - 2) \delta^{\alpha_2\alpha_3} p^{\alpha_1} - 2^m \delta^{\alpha_2\alpha_3} q^{\alpha_1} - 2^m \delta^{\alpha_2\alpha_3} r^{\alpha_1}, \\
3T_3 &= \delta^{\alpha_1\alpha_3} p^{\alpha_2} - \delta^{\alpha_2\alpha_3} p^{\alpha_1} + \delta^{\alpha_1\alpha_2} p^{\alpha_3} + 2^m \delta^{\alpha_2\alpha_1} q^{\alpha_3} + \delta^{\alpha_3\alpha_2} r^{\alpha_1} - \delta^{\alpha_3\alpha_1} r^{\alpha_2} + \delta^{\alpha_1\alpha_2} r^{\alpha_3}, \\
3T_4 &= -\delta^{\alpha_1\alpha_3} 2^m p^{\alpha_2} - \delta^{\alpha_1\alpha_3} 2^m q^{\alpha_2} + \delta^{\alpha_1\alpha_3} (2^m - 2) r^{\alpha_2}, \\
3T_5 &= \delta^{\alpha_1\alpha_3} p^{\alpha_2} - \delta^{\alpha_1\alpha_2} p^{\alpha_3} + \delta^{\alpha_3\alpha_2} p^{\alpha_1} + \delta^{\alpha_3\alpha_2} q^{\alpha_1} - \delta^{\alpha_1\alpha_2} q^{\alpha_3} + \delta^{\alpha_3\alpha_1} q^{\alpha_2} + (2 - 2^m) \delta^{\alpha_1\alpha_2} r^{\alpha_3}.
\end{aligned}$$

Thus,

$$X_l = 2^m \frac{2\pi^2}{3} i e^{\frac{i}{2} l_1 \Theta l_2} \tilde{a}_{\alpha, l} (q^{\alpha_3} \delta^{\alpha_1\alpha_2} + r^{\alpha_2} \delta^{\alpha_1\alpha_3} + p^{\alpha_1} \delta^{\alpha_2\alpha_3}) \quad (2.22)$$

and

$$\oint (\mathbb{A}^+)^3 = i \, 2c (S_1 + S_2 + S_3),$$

where S_1 , S_2 and S_3 correspond to respectively $q^{\alpha_3} \delta^{\alpha_1 \alpha_2}$, $r^{\alpha_2} \delta^{\alpha_1 \alpha_3}$ and $p^{\alpha_1} \delta^{\alpha_2 \alpha_3}$. In S_1 , we permute the l_i variables the following way: $l_1 \mapsto l_3$, $l_2 \mapsto l_1$, $l_3 \mapsto l_2$. Therefore, $l_3 \cdot \Theta l_1 \mapsto l_3 \cdot \Theta l_1$ and $q \mapsto r$. With a similar permutation of the α_i , we see that $S_1 = S_2$. We apply the same principles to prove that $S_1 = S_3$ (using permutation $l_1 \mapsto l_2$, $l_2 \mapsto l_3$, $l_3 \mapsto l_1$). Thus,

$$\frac{1}{3} \oint (\mathbb{A}^+)^3 = i \, 2c \sum_{l_i} \tilde{a}_{\alpha, l} e^{\frac{i}{2} l_1 \cdot \Theta l_2} (l_1 - l_2)^{\alpha_3} \delta^{\alpha_1 \alpha_2} = S_4 - S_5,$$

where S_4 correspond to l_1 and S_5 to l_2 . We permute the l_i variables in S_5 the following way: $l_1 \mapsto l_2$, $l_2 \mapsto l_1$, $l_3 \mapsto l_3$, with a similar permutation on the α_i . Since $l_1 \cdot \Theta l_2 \mapsto -l_1 \cdot \Theta l_2$, we finally get

$$\frac{1}{3} \oint (\mathbb{A}^+)^3 = -4c \sum_{l_i} a_{\alpha_1, l_1} a_{\alpha_2, l_2} a_{\alpha_3, -l_1 - l_2} \sin \frac{l_1 \cdot \Theta l_2}{2} l_1^{\alpha_3} \delta^{\alpha_1 \alpha_2}.$$

(iii) Lemma 2.3.11 entails that $\oint \mathbb{A}^{++++} = \text{Res}_{s=0} \sum_{(l_1, l_2, l_3) \in (\mathbb{Z}^n)^3} f_{\mu, \alpha}(s, l) \text{Tr} \gamma^{\mu, \alpha}$ where

$$\begin{aligned} \theta &:= l_1 \cdot \Theta l_2 + l_1 \cdot \Theta l_3 + l_2 \cdot \Theta l_3, \\ \text{Tr} \gamma^{\mu, \alpha} &:= \text{Tr}(\gamma^{\alpha_4} \gamma^{\mu_4} \gamma^{\alpha_3} \gamma^{\mu_3} \gamma^{\alpha_2} \gamma^{\mu_2} \gamma^{\alpha_1} \gamma^{\mu_1}), \\ f_{\mu, \alpha}(s, l) &:= \sum_{k \in \mathbb{Z}^n} ' e^{\frac{i}{2} \theta} \frac{k_{\mu_1} (k+l_1)_{\mu_2} (k+l_2)_{\mu_3} (k+l_3)_{\mu_4}}{|k|^{s+2} |k+l_1|^2 |k+l_1+l_2|^2 |k+l_1+l_2+l_3|^2} \tilde{a}_{\alpha, l}, \\ \tilde{a}_{\alpha, l} &:= a_{\alpha_1, l_1} a_{\alpha_2, l_2} a_{\alpha_3, l_3} a_{\alpha_4, -l_1 - l_2 - l_3}. \end{aligned}$$

Using (2.37) and Corollary 2.4.12 successively, we find

$$f_{\mu, \alpha}(s, l) \sim \sum_{k \in \mathbb{Z}^n} ' e^{\frac{i}{2} \theta} \frac{k_{\mu_1} k_{\mu_2} k_{\mu_3} k_{\mu_4}}{|k|^{s+2} |k+l_1|^2 |k+l_1+l_2|^2 |k+l_1+l_2+l_3|^2} \tilde{a}_{\alpha, l} \sim \sum_{k \in \mathbb{Z}^n} ' e^{\frac{i}{2} \theta} \frac{k_{\mu_1} k_{\mu_2} k_{\mu_3} k_{\mu_4}}{|k|^{s+8}} \tilde{a}_{\alpha, l}.$$

Since the function $\sum_{k \in \mathbb{Z}^n} ' e^{\frac{i}{2} \theta} \frac{k_{\mu_1} k_{\mu_2} k_{\mu_3} k_{\mu_4}}{|k|^{s+8}} \tilde{a}_{\alpha, l}$ satisfies (H2), Lemma 2.4.13 entails that

$$\oint (\mathbb{A}^+)^4 = \sum_{(l_1, l_2, l_3) \in (\mathbb{Z}^n)^3} e^{\frac{i}{2} \theta} \tilde{a}_{\alpha, l} \text{Res}_{s=0} \sum_{k \in \mathbb{Z}^n} ' \frac{k_{\mu_1} k_{\mu_2} k_{\mu_3} k_{\mu_4}}{|k|^{s+8}} \text{Tr} \gamma^{\mu, \alpha} =: \sum_l X_l.$$

Therefore, with (2.40), we get $X_l = \frac{\pi^2}{12} \tilde{a}_{\alpha, l} e^{\frac{i}{2} \theta} (A + B + C)$, where

$$\begin{aligned} A &:= \text{Tr}(\gamma^{\alpha_4} \gamma^{\mu_4} \gamma^{\alpha_3} \gamma_{\mu_4} \gamma^{\alpha_2} \gamma^{\mu_2} \gamma^{\alpha_1} \gamma_{\mu_2}), \\ B &:= \text{Tr}(\gamma^{\alpha_4} \gamma^{\mu_4} \gamma^{\alpha_3} \gamma^{\mu_2} \gamma^{\alpha_2} \gamma_{\mu_4} \gamma^{\alpha_1} \gamma_{\mu_2}), \\ C &:= \text{Tr}(\gamma^{\alpha_4} \gamma^{\mu_4} \gamma^{\alpha_3} \gamma_{\mu_2} \gamma^{\alpha_2} \gamma^{\mu_2} \gamma^{\alpha_1} \gamma_{\mu_4}). \end{aligned}$$

Using successively $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$ and $\gamma^\mu \gamma_\mu = 2^m \, 1_{2^m}$, we see that

$$\begin{aligned} A &= C = 4 \, \text{Tr}(\gamma^{\alpha_4} \gamma^{\alpha_3} \gamma^{\alpha_2} \gamma^{\alpha_1}), \\ B &= -4 \, (\text{Tr}(\gamma^{\alpha_4} \gamma^{\alpha_3} \gamma^{\alpha_1} \gamma^{\alpha_2}) + \text{Tr}(\gamma^{\alpha_4} \gamma^{\alpha_2} \gamma^{\alpha_3} \gamma^{\alpha_1})). \end{aligned}$$

Thus, $A + B + C = 8 \, 2^m (\delta^{\alpha_4 \alpha_3} \delta^{\alpha_2 \alpha_1} + \delta^{\alpha_4 \alpha_1} \delta^{\alpha_3 \alpha_2} - 2 \delta^{\alpha_4 \alpha_2} \delta^{\alpha_3 \alpha_1})$, and

$$X_l = \frac{2\pi^2}{3} \, 2^m \, e^{\frac{i}{2}\theta} \tilde{a}_{\alpha,l} (\delta^{\alpha_4 \alpha_3} \delta^{\alpha_2 \alpha_1} + \delta^{\alpha_4 \alpha_1} \delta^{\alpha_3 \alpha_2} - 2 \delta^{\alpha_4 \alpha_2} \delta^{\alpha_3 \alpha_1}). \quad (2.23)$$

By (2.23), we get

$$\oint (\mathbb{A}^+)^4 = 2c \, (-2T_1 + T_2 + T_3),$$

where

$$\begin{aligned} T_1 &:= \sum_{l_1, \dots, l_4} a_{\alpha_4, l_4} a_{\alpha_3, l_3} a_{\alpha_2, l_2} a_{\alpha_1, l_1} e^{\frac{i}{2}\theta} \delta_{0, \sum_i l_i} \delta^{\alpha_4 \alpha_2} \delta^{\alpha_3 \alpha_1}, \\ T_2 &:= \sum_{l_1, \dots, l_4} a_{\alpha_4, l_4} a_{\alpha_3, l_3} a_{\alpha_2, l_2} a_{\alpha_1, l_1} e^{\frac{i}{2}\theta} \delta_{0, \sum_i l_i} \delta^{\alpha_4 \alpha_3} \delta^{\alpha_2 \alpha_1}, \\ T_3 &:= \sum_{l_1, \dots, l_4} a_{\alpha_4, l_4} a_{\alpha_3, l_3} a_{\alpha_2, l_2} a_{\alpha_1, l_1} e^{\frac{i}{2}\theta} \delta_{0, \sum_i l_i} \delta^{\alpha_4 \alpha_1} \delta^{\alpha_3 \alpha_2}. \end{aligned}$$

We now proceed to the following permutations of the l_i variables in the T_1 term : $l_1 \mapsto l_2$, $l_2 \mapsto l_1$, $l_3 \mapsto l_4$, $l_4 \mapsto l_3$. While $\sum_i l_i$ is invariant, θ is modified : $\theta \mapsto l_2 \cdot \Theta l_1 + l_2 \cdot \Theta l_4 + l_1 \cdot \Theta l_4$. With $\delta_{0, \sum_i l_i}$ in factor, we can let l_4 be $-l_1 - l_2 - l_3$, so that $\theta \mapsto -\theta$. We also permute the α_i in the same way. Thus,

$$T_1 = \sum_{l_1, \dots, l_4} a_{\alpha_3, l_3} a_{\alpha_4, l_4} a_{\alpha_1, l_1} a_{\alpha_2, l_2} e^{-\frac{i}{2}\theta} \delta_{0, \sum_i l_i} \delta^{\alpha_3 \alpha_1} \delta^{\alpha_4 \alpha_2}.$$

Therefore,

$$2T_1 = 2 \sum_{l_1, \dots, l_4} a_{\alpha_4, l_4} a_{\alpha_3, l_3} a_{\alpha_2, l_2} a_{\alpha_1, l_1} \cos \frac{\theta}{2} \delta_{0, \sum_i l_i} \delta^{\alpha_4 \alpha_2} \delta^{\alpha_3 \alpha_1}. \quad (2.24)$$

The same principles are applied to T_2 and T_3 . Namely, the permutation $l_1 \mapsto l_1$, $l_2 \mapsto l_3$, $l_3 \mapsto l_2$, $l_4 \mapsto l_4$ in T_2 and the permutation $l_1 \mapsto l_2$, $l_2 \mapsto l_3$, $l_3 \mapsto l_1$, $l_4 \mapsto l_4$ in T_3 (the α_i variables are permuted the same way) give

$$\begin{aligned} T_2 &= \sum_{l_1, \dots, l_4} a_{\alpha_4, l_4} a_{\alpha_3, l_3} a_{\alpha_2, l_2} a_{\alpha_1, l_1} e^{\frac{i}{2}\phi} \delta_{0, \sum_i l_i} \delta^{\alpha_4 \alpha_2} \delta^{\alpha_3 \alpha_1}, \\ T_3 &= \sum_{l_1, \dots, l_4} a_{\alpha_4, l_4} a_{\alpha_3, l_3} a_{\alpha_2, l_2} a_{\alpha_1, l_1} e^{-\frac{i}{2}\phi} \delta_{0, \sum_i l_i} \delta^{\alpha_4 \alpha_2} \delta^{\alpha_3 \alpha_1} \end{aligned}$$

where $\phi := l_1 \cdot \Theta l_2 + l_1 \cdot \Theta l_3 - l_2 \cdot \Theta l_3$. Finally, we get

$$\begin{aligned} \oint (\mathbb{A}^+)^4 &= 4c \sum_{l_1, \dots, l_4} a_{\alpha_1, l_4} a_{\alpha_2, l_3} a_{l_2}^{\alpha_1} a_{l_1}^{\alpha_2} \delta_{0, \sum_i l_i} (\cos \frac{\phi}{2} - \cos \frac{\theta}{2}) \\ &= 8c \sum_{l_1, \dots, l_3} a_{\alpha_1, -l_1-l_2-l_3} a_{\alpha_2, l_3} a_{l_2}^{\alpha_1} a_{l_1}^{\alpha_2} \sin \frac{l_1 \cdot \Theta (l_2 + l_3)}{2} \sin \frac{l_2 \cdot \Theta l_3}{2}. \end{aligned} \quad (2.25)$$

(iv) Suppose $q = 2$. By Lemma 2.3.11, we get

$$\oint \mathbb{A}^\sigma = \text{Res}_{s=0} \sum_{l \in \mathbb{Z}^n} \lambda_\sigma f_{\alpha, \mu}(s, l) \text{Tr}(\gamma^{\alpha_2} \gamma^{\mu_2} \gamma^{\alpha_1} \gamma^{\mu_1})$$

where

$$f_{\alpha,\mu}(s,l) := \sum'_{k \in \mathbb{Z}^n} \frac{k_{\mu_1}(k+l)_{\mu_2}}{|k|^{s+2}|k+l|^2} e^{i\eta k \cdot \Theta l} \tilde{a}_{\alpha,l}$$

and $\eta := \frac{1}{2}(\sigma_1 - \sigma_2) \in \{-1, 1\}$. As in the proof of (i), since the presence of the phase does not change the fact that $r(s, l)$ satisfies (H1), we get

$$f_{\alpha,\mu}(s, l) \sim f_1(s, l) - f_2(s, l) + f_3(s, l)$$

where

$$\begin{aligned} f_1(s, l) &= \sum'_{k \in \mathbb{Z}^n} \frac{k_{\mu_1}(k+l)_{\mu_2}}{|k|^{s+4}} e^{i\eta k \cdot \Theta l} \tilde{a}_{\alpha,l}, \\ f_2(s, l) &= \sum'_{k \in \mathbb{Z}^n} \frac{k_{\mu_1}(k+l)_{\mu_2}(2k \cdot l + |l|^2)}{|k|^{s+6}} e^{i\eta k \cdot \Theta l} \tilde{a}_{\alpha,l}, \\ f_3(s, l) &= \sum'_{k \in \mathbb{Z}^n} \frac{k_{\mu_1}(k+l)_{\mu_2}(2k \cdot l + |l|^2)^2}{|k|^{s+8}} e^{i\eta k \cdot \Theta l} \tilde{a}_{\alpha,l}. \end{aligned}$$

Suppose that $l = 0$. Then $f_2(s, 0) = f_3(s, 0) = 0$ and Proposition 2.4.1 entails that

$$f_1(s, 0) = \sum'_{k \in \mathbb{Z}^n} \frac{k_{\mu_1} k_{\mu_2}}{|k|^{s+4}} \tilde{a}_{\alpha,0}$$

is holomorphic at 0 and so is $f_{\alpha,\mu}(s, 0)$.

Since $\frac{1}{2\pi}\Theta$ is diophantine, Theorem 2.4.4 3 gives us the result.

Suppose $q = 3$. Then Lemma 2.3.11 implies that

$$\oint \mathbb{A}^\sigma = \text{Res}_{s=0} \sum_{l \in (\mathbb{Z}^n)^2} f_{\mu,\alpha}(s, l) \text{Tr}(\gamma^{\mu_3} \gamma^{\alpha_3} \dots \gamma^{\mu_1} \gamma^{\alpha_1})$$

where

$$f_{\mu,\alpha}(s, l) := \sum'_{k \in \mathbb{Z}^n} \lambda_\sigma e^{ik \cdot \Theta(\varepsilon_1 l_1 + \varepsilon_2 l_2)} e^{\frac{i}{2} \sigma_2 l_1 \cdot \Theta l_2} \frac{k_{\mu_1}(k+l_1)_{\mu_2}(k+l_1+l_2)_{\mu_3}}{|k|^{s+2}|k+l_1|^2|k+l_1+l_2|^2} \tilde{a}_{\alpha,l},$$

and $\varepsilon_i := \frac{1}{2}(\sigma_i - \sigma_3) \in \{-1, 0, 1\}$. By hypothesis $(\varepsilon_1, \varepsilon_2) \neq (0, 0)$. There are six possibilities for the values of $(\varepsilon_1, \varepsilon_2)$, corresponding to the six possibilities for the values of σ : $(-, -, +)$, $(-, +, +)$, $(+, -, +)$, $(+, +, -)$, $(-, +, -)$, and $(+, -, -)$. As in (ii), we see that

$$\begin{aligned} f_{\mu,\alpha}(s, l) &\sim \left(\sum'_{k \in \mathbb{Z}^n} \frac{e^{ik \cdot \Theta(\varepsilon_1 l_1 + \varepsilon_2 l_2)} k_{\mu_1}(k+l_1)_{\mu_2}(k+\widehat{l}_2)_{\mu_3}}{|k|^{s+6}} \right. \\ &\quad \left. - \sum'_{k \in \mathbb{Z}^n} \frac{e^{ik \cdot \Theta(\varepsilon_1 l_1 + \varepsilon_2 l_2)} k_{\mu_1}(k+l_1)_{\mu_2}(k+\widehat{l}_2)_{\mu_3}(2k \cdot l_1 + 2k \cdot \widehat{l}_2)}{|k|^{s+8}} \lambda_\sigma \tilde{a}_{\alpha,l} e^{\frac{i}{2} \sigma_2 l_1 \cdot \Theta l_2} \right). \end{aligned}$$

With $Z := \{(l_1, l_2) : \varepsilon_1 l_1 + \varepsilon_2 l_2 = 0\}$, Theorem 2.4.4 (iii) entails that $\sum_{l \in (\mathbb{Z}^n)^2 \setminus Z} f_{\mu,\alpha}(s, l)$ is holomorphic at 0. To conclude we need to prove that

$$\sum_{\sigma} g(\sigma) := \sum_{\sigma} \sum_{l \in Z} f_{\mu,\alpha}(s, l) \text{Tr}(\gamma^{\mu_3} \gamma^{\alpha_3} \dots \gamma^{\mu_1} \gamma^{\alpha_1})$$

is holomorphic at 0. By definition, $\lambda_\sigma = i\sigma_1\sigma_2\sigma_3$ and as a consequence, we check that

$$g(-, -, +) = -g(+, +, -), \quad g(+, -, +) = -g(+, -, -), \quad g(-, +, +) = -g(-, +, -),$$

which implies that $\sum_{\sigma} g(\sigma) = 0$. The result follows.

Suppose finally that $q = 4$. Again, Lemma 2.3.11 implies that

$$\oint \mathbb{A}^{\sigma} = \operatorname{Res}_{s=0} \sum_{l \in (\mathbb{Z}^n)^3} f_{\mu, \alpha}(s, l) \operatorname{Tr}(\gamma^{\mu_4} \gamma^{\alpha_4} \dots \gamma^{\mu_1} \gamma^{\alpha_1})$$

where

$$f_{\mu, \alpha}(s, l) := \sum'_{k \in \mathbb{Z}^n} \lambda_{\sigma} e^{ik \cdot \Theta \sum_{i=1}^3 \varepsilon_i l_i} e^{\frac{i}{2}(\sigma_2 l_1 \cdot \Theta l_2 + \sigma_3(l_1 + l_2) \cdot \Theta l_3)} \frac{k_{\mu_1}(k+l_1)_{\mu_2}(k+l_1+l_2)_{\mu_3}(k+l_1+l_2+l_3)_{\mu_4}}{|k|^{s+2}|k+l_1|^2|k+l_1+l_2|^2|k+l_1+l_2+l_3|^2} \tilde{a}_{\alpha, l}$$

and $\varepsilon_i := \frac{1}{2}(\sigma_i - \sigma_4) \in \{-1, 0, 1\}$. By hypothesis $(\varepsilon_1, \varepsilon_2, \varepsilon_3) \neq (0, 0, 0)$. There are fourteen possibilities for the values of $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, corresponding to the fourteen possibilities for the values of σ : $(-, -, -, +)$, $(-, -, +, +)$, $(-, +, -, +)$, $(+, -, -, +)$, $(-, +, +, +)$, $(+, -, +, +)$, $(+, +, -, +)$, $(+, +, +, -)$, $(-, -, +, -)$, $(-, +, -, -)$, $(+, -, -, -)$, $(-, +, +, -)$, $(+, -, +, -)$ and $(+, +, -, -)$. As in (ii), we see that, with the shorthand $\theta_{\sigma} := \sigma_2 l_1 \cdot \Theta l_2 + \sigma_3(l_1 + l_2) \cdot \Theta l_3$,

$$f_{\mu, \alpha}(s, l) \sim \sum'_{k \in \mathbb{Z}^n} \lambda_{\sigma} e^{ik \cdot \Theta \sum_{i=1}^3 \varepsilon_i l_i} e^{\frac{i}{2}\theta_{\sigma}} \frac{k_{\mu_1} k_{\mu_2} k_{\mu_3} k_{\mu_4}}{|k|^{s+8}} \tilde{a}_{\alpha, l} =: g_{\mu, \alpha}(s, l).$$

With $Z_{\sigma} := \{(l_1, l_2, l_3) : \sum_{i=1}^3 \varepsilon_i l_i = 0\}$, Theorem 2.4.4 (iii), the series $\sum_{l \in (\mathbb{Z}^n)^3 \setminus Z_{\sigma}} f_{\mu, \alpha}(s, l)$ is holomorphic at 0. To conclude, we need to prove that

$$\sum_{\sigma} g(\sigma) := \sum_{\sigma} \operatorname{Res}_{s=0} \sum_{l \in Z_{\sigma}} g_{\mu, \alpha}(s, l) \operatorname{Tr}(\gamma^{\mu_4} \gamma^{\alpha_4} \dots \gamma^{\mu_1} \gamma^{\alpha_1}) = 0.$$

Let C be the set of the fourteen values of σ and C_7 be the set of the seven first values of σ given above. Lemma 2.3.7 implies

$$\sum_{\sigma \in C} g(\sigma) = 2 \sum_{\sigma \in C_7} g(\sigma).$$

Thus, in the following, we restrict to these seven values. Let us note $F_{\mu}(s) := \sum'_{k \in \mathbb{Z}^n} \frac{k_{\mu_1} k_{\mu_2} k_{\mu_3} k_{\mu_4}}{|k|^{s+8}}$ so that

$$g(\sigma) = \operatorname{Res}_{s=0} F_{\mu}(s) \lambda_{\sigma} \sum_{l \in Z_{\sigma}} e^{\frac{i}{2}\theta_{\sigma}} \tilde{a}_{\alpha, l} \operatorname{Tr}(\gamma^{\mu_4} \gamma^{\alpha_4} \dots \gamma^{\mu_1} \gamma^{\alpha_1}).$$

Recall from (2.23) that

$$\operatorname{Res}_{s=0} F_{\mu}(s) \operatorname{Tr}(\gamma^{\mu_4} \gamma^{\alpha_4} \dots \gamma^{\mu_1} \gamma^{\alpha_1}) = 2c(\delta^{\alpha_4 \alpha_3} \delta^{\alpha_2 \alpha_1} + \delta^{\alpha_4 \alpha_1} \delta^{\alpha_3 \alpha_2} - 2\delta^{\alpha_4 \alpha_2} \delta^{\alpha_3 \alpha_1}).$$

As a consequence, we get, with $\tilde{a}_{\alpha, l} := a_{\alpha_1, l_1} \dots a_{\alpha_4, l_4}$,

$$\begin{aligned} g(\sigma) &= 2c\lambda_{\sigma} \sum_{l \in (\mathbb{Z}^n)^4} e^{\frac{i}{2}\theta_{\sigma}} \tilde{a}_{\alpha, l} \delta_{\sum_{i=1}^4 l_i, 0} \delta_{\sum_{i=1}^3 \varepsilon_i l_i, 0} (\delta^{\alpha_4 \alpha_3} \delta^{\alpha_2 \alpha_1} + \delta^{\alpha_4 \alpha_1} \delta^{\alpha_3 \alpha_2} - 2\delta^{\alpha_4 \alpha_2} \delta^{\alpha_3 \alpha_1}) \\ &=: 2c\lambda_{\sigma}(T_1 + T_2 - 2T_3). \end{aligned}$$

We proceed to the following change of variable in T_1 : $l_1 \mapsto l_1$, $l_2 \mapsto l_3$, $l_3 \mapsto l_2$, $l_4 \mapsto l_4$. Thus, we get $\theta_{\sigma} \mapsto \psi_{\sigma} := \sigma_2 l_1 \cdot \Theta l_3 + \sigma_3(l_1 + l_3) \cdot \Theta l_2$, and $\sum_{i=1}^3 \varepsilon_i l_i \mapsto \varepsilon_1 l_1 + \varepsilon_3 l_2 + \varepsilon_2 l_3 =: u_{\sigma}(l)$. With a similar permutation on the α_i , we get

$$T_1 = \sum_{l \in (\mathbb{Z}^n)^4} e^{\frac{i}{2}\psi_{\sigma}} \tilde{a}_{\alpha, l} \delta_{\sum_{i=1}^4 l_i, 0} \delta_{\varepsilon_1 l_1 + \varepsilon_3 l_2 + \varepsilon_2 l_3, 0} \delta^{\alpha_4 \alpha_2} \delta^{\alpha_3 \alpha_1}.$$

We proceed to the following change of variable in T_2 : $l_1 \mapsto l_2$, $l_2 \mapsto l_3$, $l_3 \mapsto l_1$, $l_4 \mapsto l_4$. Thus, we get $\theta_\sigma \mapsto \phi_\sigma := \sigma_2 l_2 \cdot \Theta l_3 + \sigma_3(l_2 + l_3) \cdot \Theta l_1$, and $\sum_{i=1}^3 \varepsilon_i l_i \mapsto \varepsilon_3 l_1 + \varepsilon_1 l_2 + \varepsilon_2 l_3 =: v_\sigma(l)$. After a similar permutation on the α_i , we get

$$T_2 = \sum_{l \in (\mathbb{Z}^n)^4} e^{\frac{i}{2}\phi_\sigma} \tilde{a}_{\alpha,l} \delta_{\sum_{i=1}^4 l_i,0} \delta_{\varepsilon_3 l_1 + \varepsilon_1 l_2 + \varepsilon_2 l_3,0} \delta^{\alpha_4 \alpha_2} \delta^{\alpha_3 \alpha_1}.$$

Finally, we proceed to the following change of variable in T_3 : $l_1 \mapsto l_2$, $l_2 \mapsto l_1$, $l_3 \mapsto l_4$, $l_4 \mapsto l_3$. Thus, we get $\theta_\sigma \mapsto -\theta_\sigma$, and $\sum_{i=1}^3 \varepsilon_i l_i \mapsto (\varepsilon_2 - \varepsilon_3)l_1 + (\varepsilon_1 - \varepsilon_3)l_2 - \varepsilon_3 l_3 =: w_\sigma(l)$. With a similar permutation on the α_i , we get

$$T_3 = \sum_{l \in (\mathbb{Z}^n)^4} e^{-\frac{i}{2}\theta_\sigma} \tilde{a}_{\alpha,l} \delta_{\sum_{i=1}^4 l_i,0} \delta_{(\varepsilon_2 - \varepsilon_3)l_1 + (\varepsilon_1 - \varepsilon_3)l_2 - \varepsilon_3 l_3,0} \delta^{\alpha_4 \alpha_2} \delta^{\alpha_3 \alpha_1}.$$

As a consequence, we get

$$g(\sigma) = 2c \sum_{l \in (\mathbb{Z}^n)^4} K_\sigma(l_1, l_2, l_3) \tilde{a}_{\alpha,l} \delta_{\sum_{i=1}^4 l_i,0} \delta^{\alpha_4 \alpha_2} \delta^{\alpha_3 \alpha_1},$$

where $K_\sigma(l_1, l_2, l_3) = \lambda_\sigma(e^{\frac{i}{2}\psi_\sigma} \delta_{u_\sigma(l),0} + e^{\frac{i}{2}\phi_\sigma} \delta_{v_\sigma(l),0} - e^{\frac{i}{2}\theta_\sigma} \delta_{\sum_{i=1}^3 \varepsilon_i l_i,0} - e^{-\frac{i}{2}\theta_\sigma} \delta_{w_\sigma(l),0})$.

The computation of $K_\sigma(l_1, l_2, l_3)$ for the seven values of σ yields

$$\begin{aligned} K_{--++}(l_1, l_2, l_3) &= \delta_{l_1+l_3,0} + \delta_{l_2+l_3,0} - \delta_{l_1+l_2,0} - \delta_{l_1+l_2,0}, \\ K_{-+-+}(l_1, l_2, l_3) &= \delta_{l_1+l_2,0} + \delta_{l_1+l_2,0} - \delta_{l_1+l_3,0} - \delta_{l_1+l_3,0}, \\ K_{--++}(l_1, l_2, l_3) &= \delta_{l_2+l_3,0} + \delta_{l_1+l_3,0} - \delta_{l_2+l_3,0} - \delta_{l_2+l_3,0}, \\ K_{----}(l_1, l_2, l_3) &= -(e^{\frac{i}{2}l_1 \cdot \Theta l_2} \delta_{\sum_{i=1}^3 l_i,0} + e^{\frac{i}{2}l_2 \cdot \Theta l_1} \delta_{\sum_{i=1}^3 l_i,0} - e^{\frac{i}{2}l_2 \cdot \Theta l_1} \delta_{\sum_{i=1}^3 l_i,0} - e^{\frac{i}{2}l_1 \cdot \Theta l_2} \delta_{l_3,0}), \\ K_{-+++}(l_1, l_2, l_3) &= -(e^{\frac{i}{2}l_3 \cdot \Theta l_2} \delta_{l_1,0} + e^{\frac{i}{2}l_3 \cdot \Theta l_1} \delta_{l_2,0} - e^{\frac{i}{2}l_2 \cdot \Theta l_3} \delta_{l_1,0} - e^{\frac{i}{2}l_3 \cdot \Theta l_1} \delta_{l_2,0}), \\ K_{+-++}(l_1, l_2, l_3) &= -(e^{\frac{i}{2}l_1 \cdot \Theta l_2} \delta_{l_3,0} + e^{\frac{i}{2}l_2 \cdot \Theta l_1} \delta_{l_3,0} - e^{\frac{i}{2}l_1 \cdot \Theta l_3} \delta_{l_2,0} - e^{\frac{i}{2}l_3 \cdot \Theta l_2} \delta_{l_1,0}), \\ K_{++-+}(l_1, l_2, l_3) &= -(e^{\frac{i}{2}l_1 \cdot \Theta l_3} \delta_{l_2,0} + e^{\frac{i}{2}l_2 \cdot \Theta l_3} \delta_{l_1,0} - e^{\frac{i}{2}l_1 \cdot \Theta l_2} \delta_{l_3,0} - e^{\frac{i}{2}l_2 \cdot \Theta l_1} \delta_{\sum_{i=1}^3 l_i,0}). \end{aligned}$$

Thus,

$$\sum_{\sigma \in C_7} K_\sigma(l_1, l_2, l_3) = 2i(\delta_{l_3,0} - \delta_{\sum_{i=1}^3 l_i,0}) \sin \frac{l_1 \cdot \Theta l_2}{2}$$

and

$$\sum_{\sigma \in C_7} g(\sigma) = i4c \sum_{l \in (\mathbb{Z}^n)^4} (\delta_{l_3,0} - \delta_{\sum_{i=1}^3 l_i,0}) \sin \frac{l_1 \cdot \Theta l_2}{2} \tilde{a}_{\alpha,l} \delta_{\sum_{i=1}^4 l_i,0} \delta^{\alpha_4 \alpha_2} \delta^{\alpha_3 \alpha_1}.$$

The following change of variables: $l_1 \mapsto l_2$, $l_1 \mapsto l_2$, $l_3 \mapsto l_4$, $l_4 \mapsto l_3$ gives

$$\sum_{l \in (\mathbb{Z}^n)^4} \delta_{\sum_{i=1}^3 l_i,0} \sin \frac{l_1 \cdot \Theta l_2}{2} \tilde{a}_{\alpha,l} \delta_{\sum_{i=1}^4 l_i,0} \delta^{\alpha_4 \alpha_2} \delta^{\alpha_3 \alpha_1} = - \sum_{l \in (\mathbb{Z}^n)^4} \delta_{l_3,0} \sin \frac{l_1 \cdot \Theta l_2}{2} \tilde{a}_{\alpha,l} \delta_{\sum_{i=1}^4 l_i,0} \delta^{\alpha_4 \alpha_2} \delta^{\alpha_3 \alpha_1}$$

so

$$\sum_{\sigma \in C_7} g(\sigma) = i8c \sum_{l \in (\mathbb{Z}^n)^4} \delta_{l_3,0} \sin \frac{l_1 \cdot \Theta l_2}{2} \tilde{a}_{\alpha,l} \delta_{\sum_{i=1}^4 l_i,0} \delta^{\alpha_4 \alpha_2} \delta^{\alpha_3 \alpha_1}.$$

Finally, the change of variables: $l_2 \mapsto l_4, l_4 \mapsto l_2$ gives

$$\sum_{l \in (\mathbb{Z}^n)^4} \delta_{l_3,0} \sin \frac{l_1 \cdot \Theta l_2}{2} \tilde{a}_{\alpha,l} \delta_{\sum_1^4 l_i,0} \delta^{\alpha_4 \alpha_2} \delta^{\alpha_3 \alpha_1} = - \sum_{l \in (\mathbb{Z}^n)^4} \delta_{l_3,0} \sin \frac{l_1 \cdot \Theta l_2}{2} \tilde{a}_{\alpha,l} \delta_{\sum_1^4 l_i,0} \delta^{\alpha_4 \alpha_2} \delta^{\alpha_3 \alpha_1}$$

which entails that $\sum_{\sigma \in C_7} g(\sigma) = 0$. \square

Lemma 2.3.13. *Suppose $n = 4$ and $\frac{1}{2\pi}\Theta$ diophantine. For any self-adjoint one-form A ,*

$$\zeta_{D_A}(0) - \zeta_D(0) = -c \tau(F_{\alpha_1, \alpha_2} F^{\alpha_1 \alpha_2}).$$

Proof. By (1.11) and Lemma 2.3.6 we get

$$\zeta_{D_A}(0) - \zeta_D(0) = \sum_{q=1}^n \frac{(-1)^q}{q} \sum_{\sigma \in \{+, -\}^q} \oint \mathbb{A}^\sigma.$$

By Lemma 2.3.12 (iv), we see that the crossed terms all vanish. Thus, with Lemma 2.3.7, we get

$$\zeta_{D_A}(0) - \zeta_D(0) = 2 \sum_{q=1}^n \frac{(-1)^q}{q} \oint (\mathbb{A}^+)^q. \quad (2.26)$$

By definition,

$$\begin{aligned} F_{\alpha_1 \alpha_2} &= i \sum_k (a_{\alpha_2, k} k_{\alpha_1} - a_{\alpha_1, k} k_{\alpha_2}) U_k + \sum_{k, l} a_{\alpha_1, k} a_{\alpha_2, l} [U_k, U_l] \\ &= i \sum_k [(a_{\alpha_2, k} k_{\alpha_1} - a_{\alpha_1, k} k_{\alpha_2}) - 2 \sum_l a_{\alpha_1, k-l} a_{\alpha_2, l} \sin(\frac{k \cdot \Theta l}{2})] U_k. \end{aligned}$$

Thus

$$\begin{aligned} \tau(F_{\alpha_1 \alpha_2} F^{\alpha_1 \alpha_2}) &= \sum_{\alpha_1, \alpha_2=1}^{2^m} \sum_{k \in \mathbb{Z}^4} [(a_{\alpha_2, k} k_{\alpha_1} - a_{\alpha_1, k} k_{\alpha_2}) - 2 \sum_{l' \in \mathbb{Z}^4} a_{\alpha_1, k-l'} a_{\alpha_2, l'} \sin(\frac{k \cdot \Theta l'}{2})] \\ &\quad [(a_{\alpha_2, -k} k_{\alpha_1} - a_{\alpha_1, -k} k_{\alpha_2}) - 2 \sum_{l'' \in \mathbb{Z}^4} a_{\alpha_1, -k-l''} a_{\alpha_2, l''} \sin(\frac{k \cdot \Theta l''}{2})]. \end{aligned}$$

One checks that the term in a^q of $\tau(F_{\alpha_1 \alpha_2} F^{\alpha_1 \alpha_2})$ corresponds to the term $\oint (\mathbb{A}^+)^q$ given by Lemma 2.3.12. For $q = 2$, this is

$$-2 \sum_{l \in \mathbb{Z}^4, \alpha_1, \alpha_2} a_{\alpha_1, l} a_{\alpha_2, -l} (l_{\alpha_1} l_{\alpha_2} - \delta_{\alpha_1 \alpha_2} |l|^2).$$

For $q = 3$, we compute the crossed terms:

$$i \sum_{k, k', l} (a_{\alpha_2, k} k_{\alpha_1} - a_{\alpha_1, k} k_{\alpha_2}) a_{k'}^{\alpha_1} a_l^{\alpha_2} (U_k [U_{k'}, l] + [U_{k'}, U_l] U_k),$$

which gives the following a^3 -term in $\tau(F_{\alpha_1 \alpha_2} F^{\alpha_1 \alpha_2})$

$$-8 \sum_{l_i} a_{\alpha_3, -l_1 - l_2} a_{l_2}^{\alpha_1} a_{\alpha_1, l_1} \sin \frac{l_1 \cdot \Theta l_2}{2} l_1^{\alpha_3}.$$

For $q = 4$, this is

$$-4 \sum_{l_i} a_{\alpha_1, -l_1 - l_2 - l_3} a_{\alpha_2, l_3} a_{l_2}^{\alpha_1} a_{l_1}^{\alpha_2} \sin \frac{l_1 \cdot \Theta(l_2 + l_3)}{2} \sin \frac{l_2 \cdot \Theta l_3}{2}$$

which corresponds to the term $\mathcal{f}(\mathbb{A}^+)^4$. We get finally,

$$\sum_{q=1}^n \frac{(-1)^q}{q} \mathcal{f}(\mathbb{A}^+)^q = -\frac{c}{2} \tau(F_{\alpha_1, \alpha_2} F^{\alpha_1 \alpha_2}). \quad (2.27)$$

Equations (2.26) and (2.27) yield the result. \square

Lemma 2.3.14. *Suppose $n = 2$. Then, with the same hypothesis as in Lemma 2.3.11,*

$$(i) \quad \mathcal{f}(\mathbb{A}^+)^2 = \mathcal{f}(\mathbb{A}^-)^2 = 0.$$

(ii) *Suppose $\frac{1}{2\pi}\Theta$ diophantine. Then*

$$\mathcal{f} \mathbb{A}^+ \mathbb{A}^- = \mathcal{f} \mathbb{A}^- \mathbb{A}^+ = 0.$$

Proof. (i) Lemma 2.3.11 entails that $\mathcal{f} \mathbb{A}^{++} = \text{Res}_{s=0} \sum_{l \in \mathbb{Z}^2} -f(s, l)$ where

$$f(s, l) := \sum'_{k \in \mathbb{Z}^2} \frac{k_{\mu_1}(k+l)_{\mu_2}}{|k|^{s+2}|k+l|^2} \tilde{a}_{\alpha, l} \text{Tr}(\gamma^{\alpha_2} \gamma^{\mu_2} \gamma^{\alpha_1} \gamma^{\mu_1}) =: f_{\mu, \alpha}(s, l) \text{Tr}(\gamma^{\alpha_2} \gamma^{\mu_2} \gamma^{\alpha_1} \gamma^{\mu_1})$$

and $\tilde{a}_{\alpha, l} := a_{\alpha_1, l} a_{\alpha_2, -l}$. This time, since $n = 2$, it is enough to apply just once (2.37) to obtain an absolutely convergent series. Indeed, we get with (2.37)

$$f_{\mu, \alpha}(s, l) = \sum'_{k \in \mathbb{Z}^2} \frac{k_{\mu_1}(k+l)_{\mu_2}}{|k|^{s+4}} \tilde{a}_{\alpha, l} - \sum'_{k \in \mathbb{Z}^2} \frac{k_{\mu_1}(k+l)_{\mu_2}(2k \cdot l + |l|^2)}{|k|^{s+4}|k+l|^2} \tilde{a}_{\alpha, l}.$$

and the function $r(s, l) := \sum'_{k \in \mathbb{Z}^2} \frac{k_{\mu_1}(k+l)_{\mu_2}(2k \cdot l + |l|^2)}{|k|^{s+4}|k+l|^2} \tilde{a}_{\alpha, l}$ is a linear combination of functions of the type $H(s, l)$ satisfying the hypothesis of Corollary 2.4.12. As a consequence, $r(s, l)$ satisfies (H1) and

$$f_{\mu, \alpha}(s, l) \sim \sum'_{k \in \mathbb{Z}^2} \frac{k_{\mu_1}(k+l)_{\mu_2}}{|k|^{s+4}} \tilde{a}_{\alpha, l} \sim \sum'_{k \in \mathbb{Z}^2} \frac{k_{\mu_1} k_{\mu_2}}{|k|^{s+4}} \tilde{a}_{\alpha, l}$$

Note that the function $(s, l) \mapsto h_{\mu, \alpha}(s, l) := \sum'_{k \in \mathbb{Z}^2} \frac{k_{\mu_1} k_{\mu_2}}{|k|^{s+4}} \tilde{a}_{\alpha, l}$ satisfies (H2). Thus, Lemma 2.4.13 yields

$$\text{Res}_{s=0} f(s, l) = \sum_{l \in \mathbb{Z}^2} \text{Res}_{s=0} h_{\mu, \alpha}(s, l) \text{Tr}(\gamma^{\alpha_2} \gamma^{\mu_2} \gamma^{\alpha_1} \gamma^{\mu_1}).$$

By Proposition 2.4.15, we get $\text{Res}_{s=0} h_{\mu, \alpha}(s, l) = \delta_{\mu_1 \mu_2} \pi \tilde{a}_{\alpha, l}$. Therefore,

$$\mathcal{f} \mathbb{A}^{++} = -\pi \sum_{l \in \mathbb{Z}^2} \tilde{a}_{\alpha, l} \text{Tr}(\gamma^{\alpha_2} \gamma^{\mu} \gamma^{\alpha_1} \gamma_{\mu}) = 0$$

according to (2.20).

(ii) By Lemma 2.3.11, we obtain that $\oint \mathbb{A}^{-+} = \text{Res}_{s=0} \sum_{l \in \mathbb{Z}^2} \lambda_\sigma f_{\alpha,\mu}(s, l) \text{Tr}(\gamma^{\alpha_2} \gamma^{\mu_2} \gamma^{\alpha_1} \gamma^{\mu_1})$ where $\lambda_\sigma = -(-i)^2 = 1$ and

$$f_{\alpha,\mu}(s, l) := \sum'_{k \in \mathbb{Z}^2} \frac{k_{\mu_1}(k+l)_{\mu_2}}{|k|^{s+2}|k+l|^2} e^{i\eta k \cdot \Theta l} \tilde{a}_{\alpha,l}$$

and $\eta := \frac{1}{2}(\sigma_1 - \sigma_2) = -1$. As in the proof of (i), since the presence of the phase does not change the fact that $r(s, l)$ satisfies (H1), we get

$$f_{\alpha,\mu}(s, l) \sim \sum'_{k \in \mathbb{Z}^2} \frac{k_{\mu_1}(k+l)_{\mu_2}}{|k|^{s+4}} e^{i\eta k \cdot \Theta l} \tilde{a}_{\alpha,l} := g_{\alpha,\mu}(s, l).$$

Since $\frac{1}{2\pi}\Theta$ is diophantine, the functions $s \mapsto \sum_{l \in \mathbb{Z}^2 \setminus \{0\}} g_{\alpha,\mu}(s, l)$ are holomorphic at $s = 0$ by Theorem 2.4.4 3. As a consequence,

$$\oint \mathbb{A}^{-+} = \text{Res}_{s=0} g_{\alpha,\mu}(s, 0) \text{Tr}(\gamma^{\alpha_2} \gamma^{\mu_2} \gamma^{\alpha_1} \gamma^{\mu_1}) = \text{Res}_{s=0} \sum'_{k \in \mathbb{Z}^2} \frac{k_{\mu_1} k_{\mu_2}}{|k|^{s+4}} \tilde{a}_{\alpha,0} \text{Tr}(\gamma^{\alpha_2} \gamma^{\mu_2} \gamma^{\alpha_1} \gamma^{\mu_1}).$$

Recall from Proposition 2.4.1 that $\text{Res}_{s=0} \sum'_{k \in \mathbb{Z}^2} \frac{k_i k_j}{|k|^{s+4}} = \delta_{ij} \pi$. Thus, again with (2.20),

$$\oint \mathbb{A}^{-+} = \tilde{a}_{\alpha,0} \pi \text{Tr}(\gamma^{\alpha_2} \gamma^{\mu} \gamma^{\alpha_1} \gamma_{\mu}) = 0. \quad \square$$

Lemma 2.3.15. *Suppose $n = 2$ and $\frac{1}{2\pi}\Theta$ diophantine. For any self-adjoint one-form A ,*

$$\zeta_{D_A}(0) - \zeta_D(0) = 0.$$

Proof. As in Lemma 2.3.13, we use (1.11) and Lemma 2.3.6 so the result follows from Lemma 2.3.14. \square

Odd dimensional case

Lemma 2.3.16. *Suppose n odd and $\frac{1}{2\pi}\Theta$ diophantine. Then for any self-adjoint 1-form A and $\sigma \in \{-, +\}^q$ with $2 \leq q \leq n$,*

$$\oint \mathbb{A}^\sigma = 0.$$

Proof. Since $\mathbb{A}^\sigma \in \Psi_1(\mathcal{A})$, Lemma 2.2.12 with $k = n$ gives the result. \square

Corollary 2.3.17. *With the same hypothesis of Lemma 2.3.16, for any self-adjoint one-form A ,* $\zeta_{D_A}(0) - \zeta_D(0) = 0$.

Proof. As in Lemma 2.3.13, we use (1.11) and Lemma 2.3.6 so the result follows from Lemma 2.3.16. \square

2.3.2 Proof of the main result

Proof of Theorem 2.3.1. (i) By (1.6) and Proposition 2.2.6, we get

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = 4\pi\Phi_2\Lambda^2 + \Phi(0)\zeta_{D_A}(0) + \mathcal{O}(\Lambda^{-2}),$$

where $\Phi_2 = \frac{1}{2} \int_0^\infty \Phi(t) dt$. By Lemma 2.3.15, $\zeta_{D_A}(0) - \zeta_D(0) = 0$ and from Proposition 2.2.5, $\zeta_D(0) = 0$, so we get the result.

(ii) Similarly, $\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = 8\pi^2\Phi_4\Lambda^4 + \Phi(0)\zeta_{D_A}(0) + \mathcal{O}(\Lambda^{-2})$ with $\Phi_4 = \frac{1}{2} \int_0^\infty \Phi(t) t dt$. Lemma 2.3.13 implies that $\zeta_{D_A}(0) - \zeta_D(0) = -c\tau(F_{\mu\nu}F^{\mu\nu})$ and by Proposition 2.2.5, $\zeta_{D_A}(0) = -c\tau(F_{\mu\nu}F^{\mu\nu})$ leading to the result.

(iii) is a direct consequence of (1.6), Propositions 2.2.5, 2.2.6, and Corollary 2.3.17. \square

2.4 Holomorphic continuation and residues of series of zeta functions

In the following, the prime in \sum' means that we omit terms with division by zero in the summand. B^n (resp. S^{n-1}) is the closed ball (resp. the sphere) of \mathbb{R}^n with center 0 and radius 1 and the Lebesgue measure on S^{n-1} will be noted dS .

For any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we denote by $|x| = \sqrt{x_1^2 + \dots + x_n^2}$ the euclidean norm and $|x|_1 := |x_1| + \dots + |x_n|$.

$\mathbb{N} = \{1, 2, \dots\}$ is the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ the set of non negative integers.

By $f(x, y) \ll_y g(x)$ uniformly in x , we mean that $|f(x, y)| \leq a(y)|g(x)|$ for all x and y for some $a(y) > 0$.

2.4.1 Residues of series and integral

In order to be able to compute later the residues of certain series, we prove here the following

Theorem 2.4.1. *Let $P(X) = \sum_{j=0}^d P_j(X) \in \mathbb{C}[X_1, \dots, X_n]$ be a polynomial function where P_j is the homogeneous part of P of degree j . The function*

$$\zeta^P(s) := \sum'_{k \in \mathbb{Z}^n} \frac{P(k)}{|k|^s}, \quad s \in \mathbb{C}$$

has a meromorphic continuation to the whole complex plane \mathbb{C} .

Moreover $\zeta^P(s)$ is not entire if and only if $\mathcal{P}_P := \{j : \int_{u \in S^{n-1}} P_j(u) dS(u) \neq 0\} \neq \emptyset$. In that case, ζ^P has only simple poles at the points $j + n$, $j \in \mathcal{P}_P$, with

$$\text{Res}_{s=j+n} \zeta^P(s) = \int_{u \in S^{n-1}} P_j(u) dS(u).$$

The proof of this theorem is based on the following lemmas.

Lemma 2.4.2. *For any polynomial $P \in \mathbb{C}[X_1, \dots, X_n]$ of total degree $\delta(P) := \sum_{i=1}^n \deg_{X_i} P$ and any $\alpha \in \mathbb{N}_0^n$, we have*

$$\partial^\alpha (P(x)|x|^{-s}) \ll_{P, \alpha, n} (1 + |s|)^{|\alpha|_1} |x|^{-\sigma - |\alpha|_1 + \delta(P)}$$

uniformly in $x \in \mathbb{R}^n$ verifying $|x| \geq 1$, where $\sigma = \Re(s)$.

Proof. By linearity, we may assume without loss of generality that $P(X) = X^\gamma$ is a monomial. It is easy to prove (for example by induction on $|\alpha|_1$) that for all $\alpha \in \mathbb{N}_0^n$ and $x \in \mathbb{R}^n \setminus \{0\}$:

$$\partial^\alpha (|x|^{-s}) = \alpha! \sum_{\substack{\beta, \mu \in \mathbb{N}_0^n \\ \beta + 2\mu = \alpha}} \binom{-s/2}{|\beta|_1 + |\mu|_1} \frac{(|\beta|_1 + |\mu|_1)!}{\beta! \mu!} \frac{x^\beta}{|x|^{\sigma + 2(|\beta|_1 + |\mu|_1)}}.$$

It follows that for all $\alpha \in \mathbb{N}_0^n$, we have uniformly in $x \in \mathbb{R}^n$ verifying $|x| \geq 1$:

$$\partial^\alpha (|x|^{-s}) \ll_{\alpha, n} (1 + |s|)^{|\alpha|_1} |x|^{-\sigma - |\alpha|_1}. \quad (2.28)$$

By Leibniz formula and (2.28), we have uniformly in $x \in \mathbb{R}^n$ verifying $|x| \geq 1$:

$$\begin{aligned} \partial^\alpha (x^\gamma |x|^{-s}) &= \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta (x^\gamma) \partial^{\alpha - \beta} (|x|^{-s}) \\ &\ll_{\gamma, \alpha, n} \sum_{\beta \leq \alpha; \beta \leq \gamma} x^{\gamma - \beta} (1 + |s|)^{|\alpha|_1 - |\beta|_1} |x|^{-\sigma - |\alpha|_1 + |\beta|_1} \\ &\ll_{\gamma, \alpha, n} (1 + |s|)^{|\alpha|_1} |x|^{-\sigma - |\alpha|_1 + |\gamma|_1}. \end{aligned} \quad \square$$

Lemma 2.4.3. *Let $P \in \mathbb{C}[X_1, \dots, X_n]$ be a polynomial of degree d . Then, the difference*

$$\Delta_P(s) := \sum'_{k \in \mathbb{Z}^n} \frac{P(k)}{|k|^s} - \int_{\mathbb{R}^n \setminus B^n} \frac{P(x)}{|x|^s} dx$$

which is defined for $\Re(s) > d + n$, extends holomorphically on the whole complex plane \mathbb{C} .

Proof. We fix in the sequel a function $\psi \in C^\infty(\mathbb{R}^n, \mathbb{R})$ verifying for all $x \in \mathbb{R}^n$

$$0 \leq \psi(x) \leq 1, \quad \psi(x) = 1 \text{ if } |x| \geq 1 \quad \text{and} \quad \psi(x) = 0 \text{ if } |x| \leq 1/2.$$

The function $f(x, s) := \psi(x) P(x) |x|^{-s}$, $x \in \mathbb{R}^n$ and $s \in \mathbb{C}$, is in $\mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{C})$ and depends holomorphically on s .

Lemma 2.4.2 above shows that f is a “gauged symbol” in the terminology of [73, p. 4]. Thus [73, Theorem 2.1] implies that $\Delta_P(s)$ extends holomorphically on the whole complex plane \mathbb{C} . However, to be complete, we will give here a short proof of Lemma 2.4.3:

It follows from the classical Euler–Maclaurin formula that for any function $h : \mathbb{R} \rightarrow \mathbb{C}$ of class \mathcal{C}^{N+1} verifying $\lim_{|t| \rightarrow +\infty} h^{(k)}(t) = 0$ and $\int_{\mathbb{R}} |h^{(k)}(t)| dt < +\infty$ for any $k = 0, \dots, N+1$, that we have

$$\sum_{k \in \mathbb{Z}} h(k) = \int_{\mathbb{R}} h(t) dt + \frac{(-1)^N}{(N+1)!} \int_{\mathbb{R}} B_{N+1}(t) h^{(N+1)}(t) dt$$

where B_{N+1} is the Bernoulli function of order $N+1$ (it is a bounded periodic function).

Fix $m' \in \mathbb{Z}^{n-1}$ and $s \in \mathbb{C}$. Applying this to the function $h(t) := \psi(m', t) P(m', t) |(m', t)|^{-s}$ (we use Lemma 2.4.2 to verify hypothesis), we obtain that for any $N \in \mathbb{N}_0$:

$$\sum_{m_n \in \mathbb{Z}} \psi(m', m_n) P(m', m_n) |(m', m_n)|^{-s} = \int_{\mathbb{R}} \psi(m', t) P(m', t) |(m', t)|^{-s} dt + \mathcal{R}_N(m'; s) \quad (2.29)$$

where $\mathcal{R}_N(m'; s) := \frac{(-1)^N}{(N+1)!} \int_{\mathbb{R}} B_{N+1}(t) \frac{\partial^{N+1}}{\partial x_n^{N+1}} (\psi(m', t) P(m', t) |(m', t)|^{-s}) dt$.

By Lemma 2.4.2,

$$\int_{\mathbb{R}} \left| B_{N+1}(t) \frac{\partial^{N+1}}{\partial x_n^{N+1}} (\psi(m', t) P(m', t) |(m', t)|^{-s}) \right| dt \ll_{P,n,N} (1 + |s|)^{N+1} (|m'| + 1)^{-\sigma - N + \delta(P)}.$$

Thus $\sum_{m' \in \mathbb{Z}^{n-1}} \mathcal{R}_N(m'; s)$ converges absolutely and define a holomorphic function in the half plane $\{\sigma = \Re(s) > \delta(P) + n - N\}$.

Since N is an arbitrary integer, by letting $N \rightarrow \infty$ and using (2.29) above, we conclude that:

$$s \mapsto \sum_{(m', m_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} \psi(m', m_n) P(m', m_n) |(m', m_n)|^{-s} - \sum_{m' \in \mathbb{Z}^{n-1}} \int_{\mathbb{R}} \psi(m', t) P(m', t) |(m', t)|^{-s} dt$$

has a holomorphic continuation to the whole complex plane \mathbb{C} .

After n iterations, we obtain that

$$s \mapsto \sum_{m \in \mathbb{Z}^n} \psi(m) P(m) |m|^{-s} - \int_{\mathbb{R}^n} \psi(x) P(x) |x|^{-s} dx$$

has a holomorphic continuation to the whole \mathbb{C} .

To finish the proof of Lemma 2.4.3, it is enough to notice that:

- $\psi(0) = 0$ and $\psi(m) = 1, \forall m \in \mathbb{Z}^n \setminus \{0\}$;
- $s \mapsto \int_{B^n} \psi(x) P(x) |x|^{-s} dx = \int_{\{x \in \mathbb{R}^n: 1/2 \leq |x| \leq 1\}} \psi(x) P(x) |x|^{-s} dx$ is a holomorphic function on \mathbb{C} . \square

Proof of Theorem 2.4.1. Using the polar decomposition of the volume form $dx = \rho^{n-1} d\rho dS$ in \mathbb{R}^n , we get for $\Re(s) > d + n$,

$$\int_{\mathbb{R}^n \setminus B^n} \frac{P_j(x)}{|x|^s} dx = \int_1^\infty \frac{\rho^{j+n-1}}{\rho^s} \int_{S^{n-1}} P_j(u) dS(u) = \frac{1}{j+n-s} \int_{S^{n-1}} P_j(u) dS(u).$$

Lemma 2.4.3 now gives the result. \square

2.4.2 Holomorphy of certain series

Before stating the main result of this section, we give first in the following some preliminaries from Diophantine approximation theory:

Let us recall Definition 2.2.3. Let $\delta > 0$. A vector $a \in \mathbb{R}^n$ is said to be δ -diophantine if there exists $c > 0$ such that $|q \cdot a - m| \geq c |q|^{-\delta}, \forall q \in \mathbb{Z}^n \setminus \{0\}$ and $\forall m \in \mathbb{Z}$. We note $\mathcal{BV}(\delta)$ the set of δ -diophantine vectors and $\mathcal{BV} := \cup_{\delta > 0} \mathcal{BV}(\delta)$ the set of diophantine vectors.

A matrix $\Theta \in \mathcal{M}_n(\mathbb{R})$ (real $n \times n$ matrices) will be said to be diophantine if there exists $u \in \mathbb{Z}^n$ such that ${}^t\Theta(u)$ is a diophantine vector of \mathbb{R}^n .

Remark. A classical result from Diophantine approximation asserts that for all $\delta > n$, the Lebesgue measure of $\mathbb{R}^n \setminus \mathcal{BV}(\delta)$ is zero (i.e almost any element of \mathbb{R}^n is δ -diophantine).

Let $\Theta \in \mathcal{M}_n(\mathbb{R})$. If its row of index i is a diophantine vector of \mathbb{R}^n (i.e. if $L_i \in \mathcal{BV}$) then ${}^t\Theta(e_i) \in \mathcal{BV}$ and thus Θ is a diophantine matrix. It follows that almost any matrix of $\mathcal{M}_n(\mathbb{R}) \approx \mathbb{R}^{n^2}$ is diophantine.

The goal of this section is to show the following

Theorem 2.4.4. *Let $P \in \mathbb{C}[X_1, \dots, X_n]$ be a homogeneous polynomial of degree d and let b be in $\mathcal{S}(\mathbb{Z}^n \times \dots \times \mathbb{Z}^n)$ (q times, $q \in \mathbb{N}$). Then,*

(i) *Let $a \in \mathbb{R}^n$. We define $f_a(s) := \sum'_{k \in \mathbb{Z}^n} \frac{P(k)}{|k|^s} e^{2\pi i k \cdot a}$.*

1. *If $a \in \mathbb{Z}^n$, then f_a has a meromorphic continuation to the whole complex plane \mathbb{C} . Moreover f_a is not entire if and only if $\int_{u \in S^{n-1}} P(u) dS(u) \neq 0$. In that case, f_a has only a simple pole at the point $d+n$, with $\text{Res}_{s=d+n} f_a(s) = \int_{u \in S^{n-1}} P(u) dS(u)$.*

2. *If $a \in \mathbb{R}^n \setminus \mathbb{Z}^n$, then $f_a(s)$ extends holomorphically to the whole complex plane \mathbb{C} .*

(ii) *Suppose that $\Theta \in \mathcal{M}_n(\mathbb{R})$ is diophantine. For any $(\varepsilon_i)_i \in \{-1, 0, 1\}^q$, the function*

$$g(s) := \sum_{l \in (\mathbb{Z}^n)^q} b(l) f_{\Theta \sum_i \varepsilon_i l_i}(s)$$

extends meromorphically to the whole complex plane \mathbb{C} with only one possible pole on $s = d+n$.

Moreover, if we set $\mathcal{Z} := \{l \in (\mathbb{Z}^n)^q : \sum_{i=1}^q \varepsilon_i l_i = 0\}$ and $V := \sum_{l \in \mathcal{Z}} b(l)$, then

1. *If $V \int_{S^{n-1}} P(u) dS(u) \neq 0$, then $s = d+n$ is a simple pole of $g(s)$ and*

$$\text{Res}_{s=d+n} g(s) = V \int_{u \in S^{n-1}} P(u) dS(u).$$

2. *If $V \int_{S^{n-1}} P(u) dS(u) = 0$, then $g(s)$ extends holomorphically to the whole complex plane \mathbb{C} .*

(iii) *Suppose that $\Theta \in \mathcal{M}_n(\mathbb{R})$ is diophantine. For any $(\varepsilon_i)_i \in \{-1, 0, 1\}^q$, the function*

$$g_0(s) := \sum_{l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}} b(l) f_{\Theta \sum_{i=1}^q \varepsilon_i l_i}(s)$$

where $\mathcal{Z} := \{l \in (\mathbb{Z}^n)^q : \sum_{i=1}^q \varepsilon_i l_i = 0\}$ extends holomorphically to the whole complex plane \mathbb{C} .

Proof of Theorem 2.4.4: First we remark that

If $a \in \mathbb{Z}^n$ then $f_a(s) = \sum'_{k \in \mathbb{Z}^n} \frac{P(k)}{|k|^s}$. So, the point (i.1) follows from Theorem 2.4.1;

$g(s) := \sum_{l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}} b(l) f_{\Theta \sum_i \varepsilon_i l_i}(s) + (\sum_{l \in \mathcal{Z}} b(l)) \sum'_{k \in \mathbb{Z}^n} \frac{P(k)}{|k|^s}$. Thus, the point (ii) rises

easily from (iii) and Theorem 2.4.1.

So, to complete the proof, it remains to prove the items (i.2) and (iii).

The direct proof of (i.2) is easy but is not sufficient to deduce (iii) of which the proof is more delicate and requires a more precise (i.e. more effective) version of (i.2). The next lemma gives such crucial version, but before, let us give some notations:

$$\mathcal{F} := \left\{ \frac{P(X)}{(X_1^2 + \dots + X_n^2 + 1)^{r/2}} : P(X) \in \mathbb{C}[X_1, \dots, X_n] \text{ and } r \in \mathbb{N}_0 \right\}.$$

We set $g = \deg(G) = \deg(P) - r \in \mathbb{Z}$, the degree of $G = \frac{P(X)}{(X_1^2 + \dots + X_n^2 + 1)^{r/2}} \in \mathcal{F}$.

By convention we set $\deg(0) = -\infty$.

Lemma 2.4.5. *Let $a \in \mathbb{R}^n$. We assume that $d(a.u, \mathbb{Z}) := \inf_{m \in \mathbb{Z}} |a.u - m| > 0$ for some $u \in \mathbb{Z}^n$. For all $G \in \mathcal{F}$, we define formally,*

$$F_0(G; a; s) := \sum'_{k \in \mathbb{Z}^n} \frac{G(k)}{|k|^s} e^{2\pi i k \cdot a} \quad \text{and} \quad F_1(G; a; s) := \sum_{k \in \mathbb{Z}^n} \frac{G(k)}{(|k|^2 + 1)^{s/2}} e^{2\pi i k \cdot a}.$$

Then for all $N \in \mathbb{N}$, all $G \in \mathcal{F}$ and all $i \in \{0, 1\}$, there exist positive constants $C_i := C_i(G, N, u)$, $B_i := B_i(G, N, u)$ and $A_i := A_i(G, N, u)$ such that $s \mapsto F_i(G; \alpha; s)$ extends holomorphically to the half-plane $\{\Re(s) > -N\}$ and verifies in it:

$$F_i(G; a; s) \leq C_i(1 + |s|)^{B_i} (d(a.u, \mathbb{Z}))^{-A_i}.$$

Remark 2.4.6. The important point here is that we obtain an explicit bound of $F_i(G; \alpha; s)$ in $\{\Re(s) > -N\}$ which depends on the vector a only through $d(a.u, \mathbb{Z})$, so depends on u and indirectly on a (in the sequel, a will vary). In particular the constants $C_i := C_i(G, N, u)$, $B_i = B_i(G, N)$ and $A_i := A_i(G, N)$ do not depend on the vector a but only on u . This is crucial for the proof of items (ii) and (iii) of Theorem 2.4.4!

Proof of Lemma 2.4.5 for $i = 1$:

Let $N \in \mathbb{N}_0$ be a fixed integer, and set $g_0 := n + N + 1$.

We will prove Lemma 2.4.5 by induction on $g = \deg(G) \in \mathbb{Z}$. More precisely, in order to prove case $i = 1$, it suffices to prove that:

Lemma 2.4.5 is true for all $G \in \mathcal{F}$ verifying $\deg(G) \leq -g_0$.

Let $g \in \mathbb{Z}$ with $g \geq -g_0 + 1$. If Lemma 2.4.5 is true for all $G \in \mathcal{F}$ such that $\deg(G) \leq g - 1$,

then it is also true for all $G \in \mathcal{F}$ satisfying $\deg(G) = g$.

- Step 1: Checking Lemma 2.4.5 for $\deg(G) \leq -g_0 := -(n + N + 1)$.

Let $G(X) = \frac{P(X)}{(X_1^2 + \dots + X_n^2 + 1)^{r/2}} \in \mathcal{F}$ verifying $\deg(G) \leq -g_0$. It is easy to see that we have uniformly in $s = \sigma + i\tau \in \mathbb{C}$ and in $k \in \mathbb{Z}^n$:

$$\frac{|G(k) e^{2\pi i k \cdot a}|}{(|k|^2 + 1)^{\sigma/2}} = \frac{|P(k)|}{(|k|^2 + 1)^{(r+\sigma)/2}} \ll_G \frac{1}{(|k|^2 + 1)^{(r+\sigma-\deg(P))/2}} \ll_G \frac{1}{(|k|^2 + 1)^{(\sigma-\deg(G))/2}} \ll_G \frac{1}{(|k|^2 + 1)^{(\sigma+g_0)/2}}.$$

It follows that $F_1(G; a; s) = \sum_{k \in \mathbb{Z}^n} \frac{G(k)}{(|k|^2 + 1)^{s/2}} e^{2\pi i k \cdot a}$ converges absolutely and defines a holomorphic function in the half plane $\{\sigma > -N\}$. Therefore, we have for any $s \in \{\Re(s) > -N\}$:

$$|F_1(G; a; s)| \ll_G \sum_{k \in \mathbb{Z}^n} \frac{1}{(|k|^2 + 1)^{(-N+g_0)/2}} \ll_G \sum_{k \in \mathbb{Z}^n} \frac{1}{(|k|^2 + 1)^{(n+1)/2}} \ll_G 1.$$

Thus, Lemma 2.4.5 is true when $\deg(G) \leq -g_0$.

- Step 2: Induction.

Now let $g \in \mathbb{Z}$ satisfying $g \geq -g_0 + 1$ and suppose that Lemma 2.4.5 is valid for all $G \in \mathcal{F}$ verifying $\deg(G) \leq g - 1$. Let $G \in \mathcal{F}$ with $\deg(G) = g$. We will prove that G also verifies conclusions of Lemma 2.4.5:

There exist $P \in \mathbb{C}[X_1, \dots, X_n]$ of degree $d \geq 0$ and $r \in \mathbb{N}_0$ such that $G(X) = \frac{P(X)}{(X_1^2 + \dots + X_n^2 + 1)^{r/2}}$ and $g = \deg(G) = d - r$.

Since $G(k) \ll (|k|^2 + 1)^{g/2}$ uniformly in $k \in \mathbb{Z}^n$, we deduce that $F_1(G; a; s)$ converges absolutely in $\{\sigma = \Re(s) > n + g\}$.

Since $k \mapsto k + u$ is a bijection from \mathbb{Z}^n into \mathbb{Z}^n , it follows that we also have for $\Re(s) > n + g$

$$\begin{aligned}
F_1(G; a; s) &= \sum_{k \in \mathbb{Z}^n} \frac{P(k)}{(|k|^2 + 1)^{(s+r)/2}} e^{2\pi i k \cdot a} = \sum_{k \in \mathbb{Z}^n} \frac{P(k+u)}{(|k+u|^2 + 1)^{(s+r)/2}} e^{2\pi i (k+u) \cdot a} \\
&= e^{2\pi i u \cdot a} \sum_{k \in \mathbb{Z}^n} \frac{P(k+u)}{(|k|^2 + 2k \cdot u + |u|^2 + 1)^{(s+r)/2}} e^{2\pi i k \cdot a} \\
&= e^{2\pi i u \cdot a} \sum_{\alpha \in \mathbb{N}_0^n; |\alpha|_1 = \alpha_1 + \dots + \alpha_n \leq d} \frac{u^\alpha}{\alpha!} \sum_{k \in \mathbb{Z}^n} \frac{\partial^\alpha P(k)}{(|k|^2 + 2k \cdot u + |u|^2 + 1)^{(s+r)/2}} e^{2\pi i k \cdot a} \\
&= e^{2\pi i u \cdot a} \sum_{|\alpha|_1 \leq d} \frac{u^\alpha}{\alpha!} \sum_{k \in \mathbb{Z}^n} \frac{\partial^\alpha P(k)}{(|k|^2 + 1)^{(s+r)/2}} \left(1 + \frac{2k \cdot u + |u|^2}{(|k|^2 + 1)}\right)^{-(s+r)/2} e^{2\pi i k \cdot a}.
\end{aligned}$$

Let $M := \sup(N + n + g, 0) \in \mathbb{N}_0$. We have uniformly in $k \in \mathbb{Z}^n$

$$\left(1 + \frac{2k \cdot u + |u|^2}{(|k|^2 + 1)}\right)^{-(s+r)/2} = \sum_{j=0}^M \binom{-(s+r)/2}{j} \frac{(2k \cdot u + |u|^2)^j}{(|k|^2 + 1)^j} + O_{M,u} \left(\frac{(1+|s|)^{M+1}}{(|k|^2 + 1)^{(M+1)/2}} \right).$$

Thus, for $\sigma = \Re(s) > n + d$,

$$\begin{aligned}
F_1(G; a; s) &= e^{2\pi i u \cdot a} \sum_{|\alpha|_1 \leq d} \frac{u^\alpha}{\alpha!} \sum_{k \in \mathbb{Z}^n} \frac{\partial^\alpha P(k)}{(|k|^2 + 1)^{(s+r)/2}} \left(1 + \frac{2k \cdot u + |u|^2}{(|k|^2 + 1)}\right)^{-(s+r)/2} e^{2\pi i k \cdot a} \\
&= e^{2\pi i u \cdot a} \sum_{|\alpha|_1 \leq d} \sum_{j=0}^M \frac{u^\alpha}{\alpha!} \binom{-(s+r)/2}{j} \sum_{k \in \mathbb{Z}^n} \frac{\partial^\alpha P(k) (2k \cdot u + |u|^2)^j}{(|k|^2 + 1)^{(s+r+2j)/2}} e^{2\pi i k \cdot a} \\
&\quad + O_{G,M,u} \left((1+|s|)^{M+1} \sum_{k \in \mathbb{Z}^n} \frac{1}{(|k|^2 + 1)^{(\sigma+M+1-g)/2}} \right). \tag{2.30}
\end{aligned}$$

Set $I := \{(\alpha, j) \in \mathbb{N}_0^n \times \{0, \dots, M\} \mid |\alpha|_1 \leq d\}$ and $I^* := I \setminus \{(0, 0)\}$.

Set also $G_{(\alpha,j);u}(X) := \frac{\partial^\alpha P(X) (2X \cdot u + |u|^2)^j}{(|X|^2 + 1)^{(r+2j)/2}} \in \mathcal{F}$ for all $(\alpha, j) \in I^*$.

Since $M \geq N + n + g$, it follows from (2.30) that

$$(1 - e^{2\pi i u \cdot a}) F_1(G; a; s) = e^{2\pi i u \cdot a} \sum_{(\alpha,j) \in I^*} \frac{u^\alpha}{\alpha!} \binom{-(s+r)/2}{j} F_1(G_{(\alpha,j);u}; \alpha; s) + R_N(G; a; u; s) \tag{2.31}$$

where $s \mapsto R_N(G; a; u; s)$ is a holomorphic function in the half plane $\{\sigma = \Re(s) > -N\}$, in which it satisfies the bound $R_N(G; a; u; s) \ll_{G,N,u} 1$.

Moreover it is easy to see that, for any $(\alpha, j) \in I^*$,

$$\deg(G_{(\alpha,j);u}) = \deg(\partial^\alpha P) + j - (r + 2j) \leq d - |\alpha|_1 + j - (r + 2j) = g - |\alpha|_1 - j \leq g - 1.$$

Relation (2.31) and the induction hypothesis imply then that

$$(1 - e^{2\pi i u \cdot a}) F_1(G; a; s) \text{ verifies the conclusions of Lemma 2.4.5.} \tag{2.32}$$

Since $|1 - e^{2\pi i u \cdot a}| = 2|\sin(\pi u \cdot a)| \geq d(u \cdot a, \mathbb{Z})$, then (2.32) implies that $F_1(G; a; s)$ satisfies conclusions of Lemma 2.4.5. This completes the induction and the proof for $i = 1$.

Proof of Lemma 2.4.5 for $i = 0$:

Let $N \in \mathbb{N}$ be a fixed integer. Let $G(X) = \frac{P(X)}{(X_1^2 + \dots + X_n^2 + 1)^{r/2}} \in \mathcal{F}$ and $g = \deg(G) = d - r$ where $d \geq 0$ is the degree of the polynomial P . Set also $M := \sup(N + g + n, 0) \in \mathbb{N}_0$.

Since $P(k) \ll |k|^d$ for $k \in \mathbb{Z}^n \setminus \{0\}$, it follows that $F_0(G; a; s)$ and $F_1(G; a; s)$ converge absolutely in the half plane $\{\sigma = \Re(s) > n + g\}$.

Moreover, we have for $s = \sigma + i\tau \in \mathbb{C}$ verifying $\sigma > n + g$:

$$\begin{aligned}
F_0(G; a; s) &= \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \frac{G(k)}{(|k|^2 + 1)^{s/2}} e^{2\pi i k \cdot a} = \sum'_{k \in \mathbb{Z}^n} \frac{G(k)}{(|k|^2 + 1)^{s/2}} \left(1 - \frac{1}{|k|^2 + 1}\right)^{-s/2} e^{2\pi i k \cdot a} \\
&= \sum'_{k \in \mathbb{Z}^n} \sum_{j=0}^M \binom{-s/2}{j} (-1)^j \frac{G(k)}{(|k|^2 + 1)^{(s+2j)/2}} e^{2\pi i k \cdot a} \\
&\quad + O_M((1 + |s|)^{M+1} \sum'_{k \in \mathbb{Z}^n} \frac{|G(k)|}{(|k|^2 + 1)^{(\sigma+2M+2)/2}}) \\
&= \sum_{j=0}^M \binom{-s/2}{j} (-1)^j F_1(G; a; s + 2j) \\
&\quad + O_M[(1 + |s|)^{M+1} (1 + \sum'_{k \in \mathbb{Z}^n} \frac{|G(k)|}{(|k|^2 + 1)^{(\sigma+2M+2)/2}})]. \tag{2.33}
\end{aligned}$$

In addition we have uniformly in $s = \sigma + i\tau \in \mathbb{C}$ verifying $\sigma > -N$,

$$\sum'_{k \in \mathbb{Z}^n} \frac{|G(k)|}{(|k|^2 + 1)^{(\sigma+2M+2)/2}} \ll \sum'_{k \in \mathbb{Z}^n} \frac{|k|^g}{(|k|^2 + 1)^{(-N+2M+2)/2}} \ll \sum'_{k \in \mathbb{Z}^n} \frac{1}{|k|^{n+1}} < +\infty.$$

So (2.33) and Lemma 2.4.5 for $i = 1$ imply that Lemma 2.4.5 is also true for $i = 0$. This completes the proof of Lemma 2.4.5. \square

Proof of item (i.2) of Theorem 2.4.4:

Since $a \in \mathbb{R}^n \setminus \mathbb{Z}^n$, there exists $i_0 \in \{1, \dots, n\}$ such that $a_{i_0} \notin \mathbb{Z}$. In particular $d(a \cdot e_{i_0}, \mathbb{Z}) = d(a_{i_0}, \mathbb{Z}) > 0$. Therefore, a satisfies the assumption of Lemma 2.4.5 with $u = e_{i_0}$. Thus, for all $N \in \mathbb{N}$, $s \mapsto f_a(s) = F_0(P; a; s)$ has a holomorphic continuation to the half-plane $\{\Re(s) > -N\}$. It follows, by letting $N \rightarrow \infty$, that $s \mapsto f_a(s)$ has a holomorphic continuation to the whole complex plane \mathbb{C} .

Proof of item (iii) of Theorem 2.4.4:

Let $\Theta \in \mathcal{M}_n(\mathbb{R})$, $(\varepsilon_i)_i \in \{-1, 0, 1\}^q$ and $b \in \mathcal{S}(\mathbb{Z}^n \times \mathbb{Z}^n)$. We assume that Θ is a diophantine matrix. Set $\mathcal{Z} := \{l = (l_1, \dots, l_q) \in (\mathbb{Z}^n)^q : \sum_i \varepsilon_i l_i = 0\}$ and $P \in \mathbb{C}[X_1, \dots, X_n]$ of degree $d \geq 0$.

It is easy to see that for $\sigma > n + d$:

$$\begin{aligned}
\sum_{l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}} |b(l)| \sum'_{k \in \mathbb{Z}^n} \frac{|P(k)|}{|k|^\sigma} |e^{2\pi i k \cdot \Theta \sum_i \varepsilon_i l_i}| &\ll_P \sum_{l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}} |b(l)| \sum'_{k \in \mathbb{Z}^n} \frac{1}{|k|^{\sigma-d}} \ll_{P, \sigma} \sum_{l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}} |b(l)| \\
&< +\infty.
\end{aligned}$$

So

$$g_0(s) := \sum_{l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}} b(l) f_{\Theta \sum_i \varepsilon_i l_i}(s) = \sum_{l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}} b(l) \sum_{k \in \mathbb{Z}^n} \frac{P(k)}{|k|^s} e^{2\pi i k \cdot \Theta \sum_i \varepsilon_i l_i}$$

converges absolutely in the half plane $\{\Re(s) > n + d\}$.

Moreover with the notations of Lemma 2.4.5, we have for all $s = \sigma + i\tau \in \mathbb{C}$ verifying $\sigma > n + d$:

$$g_0(s) = \sum_{l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}} b(l) f_{\Theta \sum_i \varepsilon_i l_i}(s) = \sum_{l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}} b(l) F_0(P; \Theta \sum_i \varepsilon_i l_i; s) \quad (2.34)$$

But Θ is diophantine, so there exists $u \in \mathbb{Z}^n$ and $\delta, c > 0$ such

$$|q \cdot {}^t \Theta u - m| \geq c(1 + |q|)^{-\delta}, \quad \forall q \in \mathbb{Z}^n \setminus \{0\}, \quad \forall m \in \mathbb{Z}.$$

We deduce that $\forall l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}$,

$$|(\Theta \sum_i \varepsilon_i l_i) \cdot u - m| = |(\sum_i \varepsilon_i l_i) \cdot {}^t \Theta u - m| \geq c(1 + |\sum_i \varepsilon_i l_i|)^{-\delta} \geq c(1 + |l|)^{-\delta}.$$

It follows that there exists $u \in \mathbb{Z}^n$, $\delta > 0$ and $c > 0$ such that

$$\forall l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}, \quad d((\Theta \sum_i \varepsilon_i l_i) \cdot u; \mathbb{Z}) \geq c(1 + |l|)^{-\delta}. \quad (2.35)$$

Therefore, for any $l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}$, the vector $a = \Theta \sum_i \varepsilon_i l_i$ verifies the assumption of Lemma 2.4.5 with the same u . Moreover δ and c in (2.35) are also independent on l .

We fix now $N \in \mathbb{N}$. Lemma 2.4.5 implies that there exist positive constants $C_0 := C_0(P, N, u)$, $B_0 := B_0(P, N, u)$ and $A_0 := A_0(P, N, u)$ such that for all $l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}$, $s \mapsto F_0(P; \Theta \sum_i \varepsilon_i l_i; s)$ extends *holomorphically* to the half plane $\{\Re(s) > -N\}$ and verifies in it the bound

$$F_0(P; \Theta \sum_i \varepsilon_i l_i; s) \leq C_0 (1 + |s|)^{B_0} d((\Theta \sum_i \varepsilon_i l_i) \cdot u; \mathbb{Z})^{-A_0}.$$

This and (2.35) imply that for any compact set K included in the half plane $\{\Re(s) > -N\}$, there exist two constants $C := C(P, N, c, \delta, u, K)$ and $D := D(P, N, c, \delta, u)$ (independent on $l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}$) such that

$$\forall s \in K \text{ and } \forall l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}, \quad F_0(P; \Theta \sum_i \varepsilon_i l_i; s) \leq C(1 + |l|)^D. \quad (2.36)$$

It follows that $s \mapsto \sum_{l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}} b(l) F_0(P; \Theta \sum_i \varepsilon_i l_i; s)$ has a holomorphic continuation to the half plane $\{\Re(s) > -N\}$.

This and (2.34) imply that $s \mapsto g_0(s) = \sum_{l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}} b(l) f_{\Theta \sum_i \varepsilon_i l_i}(s)$ has a holomorphic continuation to $\{\Re(s) > -N\}$. Since N is an arbitrary integer, by letting $N \rightarrow \infty$, it follows that $s \mapsto g_0(s)$ has a holomorphic continuation to the whole complex plane \mathbb{C} which completes the proof of the theorem. \square

Remark 2.4.7. By equation (2.32), we see that a Diophantine condition is sufficient to get Lemma 2.4.5. Our Diophantine condition appears also (in equivalent form) in Connes [27, Prop. 4.9] (see Remark 4.2 below). The following heuristic argument shows that our condition seems to be necessary in order to get the result of Theorem 2.4.4:

For simplicity we assume $n = 1$ (but the argument extends easily to any n).

Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$. We know (see this reflection formula in [52, p. 6]) that for any $l \in \mathbb{Z} \setminus \{0\}$,

$$g_{\theta l}(s) := \sum'_{k \in \mathbb{Z}} \frac{e^{2\pi i \theta l k}}{|k|^s} = \frac{\pi^{s-1/2}}{\Gamma(\frac{1-s}{2})} \Gamma(\frac{s}{2}) h_{\theta l}(1-s) \text{ where } h_{\theta l}(s) := \sum'_{k \in \mathbb{Z}} \frac{1}{|\theta l + k|^s}.$$

So, for any $(a_l) \in \mathcal{S}(\mathbb{Z})$, the existence of meromorphic continuation of $g_0(s) := \sum'_{l \in \mathbb{Z}} a_l g_{\theta l}(s)$ is equivalent to the existence of meromorphic continuation of

$$h_0(s) := \sum'_{l \in \mathbb{Z}} a_l h_{\theta l}(s) = \sum'_{l \in \mathbb{Z}} a_l \sum'_{k \in \mathbb{Z}} \frac{1}{|\theta l + k|^s}.$$

So, for at least one $\sigma_0 \in \mathbb{R}$, we must have $\frac{|a_l|}{|\theta l + k|^{\sigma_0}} = O(1)$ uniformly in $k, l \in \mathbb{Z}^*$.

It follows that for any $(a_l) \in \mathcal{S}(\mathbb{Z})$, $|\theta l + k| \gg |a_l|^{1/\sigma_0}$ uniformly in $k, l \in \mathbb{Z}^*$. Therefore, our Diophantine condition seems to be necessary.

Commutation between sum and residue

Let $p \in \mathbb{N}$. Recall that $\mathcal{S}((\mathbb{Z}^n)^p)$ is the set of the Schwartz sequences on $(\mathbb{Z}^n)^p$. In other words, $b \in \mathcal{S}((\mathbb{Z}^n)^p)$ if and only if for all $r \in \mathbb{N}_0$, $(1 + |l_1|^2 + \dots + |l_p|^2)^r |b(l_1, \dots, l_p)|^2$ is bounded on $(\mathbb{Z}^n)^p$. We note that if $Q \in \mathbb{R}[X_1, \dots, X_{np}]$ is a polynomial, $(a_j) \in \mathcal{S}(\mathbb{Z}^n)^p$, $b \in \mathcal{S}(\mathbb{Z}^n)$ and ϕ a real-valued function, then $l := (l_1, \dots, l_p) \mapsto \tilde{a}(l) b(-\widehat{l}_p) Q(l) e^{i\phi(l)}$ is a Schwartz sequence on $(\mathbb{Z}^n)^p$, where

$$\begin{aligned} \tilde{a}(l) &:= a_1(l_1) \cdots a_p(l_p), \\ \widehat{l}_i &:= l_1 + \dots + l_i. \end{aligned}$$

In the following, we will use several times the fact that for any $(k, l) \in (\mathbb{Z}^n)^2$ such that $k \neq 0$ and $k \neq -l$, we have

$$\frac{1}{|k + l|^2} = \frac{1}{|k|^2} - \frac{2k \cdot l + |l|^2}{|k|^2 |k + l|^2}. \quad (2.37)$$

Lemma 2.4.8. *There exists a polynomial $P \in \mathbb{R}[X_1, \dots, X_p]$ of degree $4p$ and with positive coefficients such that for any $k \in \mathbb{Z}^n$, and $l := (l_1, \dots, l_p) \in (\mathbb{Z}^n)^p$ such that $k \neq 0$ and $k \neq -\widehat{l}_i$ for all $1 \leq i \leq p$, the following holds:*

$$\frac{1}{|k + \widehat{l}_1|^2 \dots |k + \widehat{l}_p|^2} \leq \frac{1}{|k|^{2p}} P(|l_1|, \dots, |l_p|).$$

Proof. Let's fix i such that $1 \leq i \leq p$. Using two times (2.37), Cauchy-Schwarz inequality and the fact that $|k + \widehat{l}_i|^2 \geq 1$, we get

$$\begin{aligned} \frac{1}{|k + \widehat{l}_i|^2} &\leq \frac{1}{|k|^2} + \frac{2|k| |\widehat{l}_i| + |\widehat{l}_i|^2}{|k|^4} + \frac{(2|k| |\widehat{l}_i| + |\widehat{l}_i|^2)^2}{|k|^4 |k + \widehat{l}_i|^2} \\ &\leq \frac{1}{|k|^2} + \frac{2}{|k|^3} |\widehat{l}_i| + \left(\frac{1}{|k|^4} + \frac{4}{|k|^2} \right) |\widehat{l}_i|^2 + \frac{4}{|k|^3} |\widehat{l}_i|^3 + \frac{1}{|k|^4} |\widehat{l}_i|^4. \end{aligned}$$

Since $|k| \geq 1$, and $|\widehat{l}_i|^j \leq |\widehat{l}_i|^4$ if $1 \leq j \leq 4$, we find

$$\begin{aligned} \frac{1}{|k+\widehat{l}_i|^2} &\leq \frac{5}{|k|^2} \sum_{j=0}^4 |\widehat{l}_i|^j \leq \frac{5}{|k|^2} (1 + 4|\widehat{l}_i|^4) \leq \frac{5}{|k|^2} (1 + 4(\sum_{j=1}^p |l_j|)^4), \\ \frac{1}{|k+\widehat{l}_1|^2 \dots |k+\widehat{l}_p|^2} &\leq \frac{5^p}{|k|^{2p}} (1 + 4(\sum_{j=1}^p |l_j|)^4)^p. \end{aligned}$$

Taking $P(X_1, \dots, X_p) := 5^p (1 + 4(\sum_{j=1}^p X_j)^4)^p$ now gives the result. \square

Lemma 2.4.9. *Let $b \in \mathcal{S}((\mathbb{Z}^n)^p)$, $p \in \mathbb{N}$, $P_j \in \mathbb{R}[X_1, \dots, X_n]$ be a homogeneous polynomial function of degree j , $k \in \mathbb{Z}^n$, $l := (l_1, \dots, l_p) \in (\mathbb{Z}^n)^p$, $r \in \mathbb{N}_0$, ϕ be a real-valued function on $\mathbb{Z}^n \times (\mathbb{Z}^n)^p$ and*

$$h(s, k, l) := \frac{b(l) P_j(k) e^{i\phi(k, l)}}{|k|^{s+r} |k + \widehat{l}_1|^2 \dots |k + \widehat{l}_p|^2},$$

with $h(s, k, l) := 0$ if, for $k \neq 0$, one of the denominators is zero.

For all $s \in \mathbb{C}$ such that $\Re(s) > n + j - r - 2p$, the series

$$H(s) := \sum'_{(k, l) \in (\mathbb{Z}^n)^{p+1}} h(s, k, l)$$

is absolutely summable. In particular,

$$\sum'_{k \in \mathbb{Z}^n} \sum_{l \in (\mathbb{Z}^n)^p} h(s, k, l) = \sum_{l \in (\mathbb{Z}^n)^p} \sum'_{k \in \mathbb{Z}^n} h(s, k, l).$$

Proof. Let $s = \sigma + i\tau \in \mathbb{C}$ such that $\sigma > n + j - r - 2p$. By Lemma 2.4.8 we get, for $k \neq 0$,

$$|h(s, k, l)| \leq |b(l) P_j(k)| |k|^{-r-\sigma-2p} P(l),$$

where $P(l) := P(|l_1|, \dots, |l_p|)$ and P is a polynomial of degree $4p$ with positive coefficients. Thus, $|h(s, k, l)| \leq F(l) G(k)$ where $F(l) := |b(l)| P(l)$ and $G(k) := |P_j(k)| |k|^{-r-\sigma-2p}$. The summability of $\sum_{l \in (\mathbb{Z}^n)^p} F(l)$ is implied by the fact that $b \in \mathcal{S}((\mathbb{Z}^n)^p)$. The summability of $\sum'_{k \in \mathbb{Z}^n} G(k)$ is a consequence of the fact that $\sigma > n + j - r - 2p$. Finally, as a product of two summable series, $\sum_{k, l} F(l) G(k)$ is a summable series, which proves that $\sum_{k, l} h(s, k, l)$ is also absolutely summable. \square

Definition 2.4.10. Let f be a function on $D \times (\mathbb{Z}^n)^p$ where D is an open neighborhood of 0 in \mathbb{C} .

We say that f satisfies (H1) if and only if there exists $\rho > 0$ such that

(i) for any l , $s \mapsto f(s, l)$ extends as a holomorphic function on U_ρ , where U_ρ is the open disk of center 0 and radius ρ ,

(ii) the series $\sum_{l \in (\mathbb{Z}^n)^p} \|H(\cdot, l)\|_{\infty, \rho}$ is a summable series, where $\|H(\cdot, l)\|_{\infty, \rho} := \sup_{s \in U_\rho} |H(s, l)|$.

We say that f satisfies (H2) if and only if there exists $\rho > 0$ such that

(i) for any l , $s \mapsto f(s, l)$ extends as a holomorphic function on $U_\rho - \{0\}$,

(ii) for any δ such that $0 < \delta < \rho$, the series $\sum_{l \in (\mathbb{Z}^n)^p} \|H(\cdot, l)\|_{\infty, \delta, \rho}$ is summable, where $\|H(\cdot, l)\|_{\infty, \delta, \rho} := \sup_{\delta < |s| < \rho} |H(s, l)|$.

Remark 2.4.11. Note that (H1) implies (H2). Moreover, if f satisfies (H1) (resp. (H2)) for $\rho > 0$, then it is straightforward to check that $f : s \mapsto \sum_{l \in (\mathbb{Z}^n)^p} f(s, l)$ extends as an holomorphic function on U_ρ (resp. on $U_\rho \setminus \{0\}$).

Corollary 2.4.12. With the same notations of Lemma 2.4.9, suppose that $r + 2p - j > n$, then, the function $H(s, l) := \sum'_{k \in \mathbb{Z}^n} h(s, k, l)$ satisfies (H1).

Proof. (i) Let's fix $\rho > 0$ such that $\rho < r + 2p - j - n$. Since $r + 2p - j > n$, U_ρ is inside the half-plane of absolute convergence of the series defined by $H(s, l)$. Thus, $s \mapsto H(s, l)$ is holomorphic on U_ρ .

(ii) Since $||k|^{-s}| \leq |k|^\rho$ for all $s \in U_\rho$ and $k \in \mathbb{Z}^n \setminus \{0\}$, we get as in the above proof

$$|h(s, k, l)| \leq |b(l) P_j(k)| |k|^{-r+\rho-2p} P(|l_1|, \dots, |l_p|).$$

Since $\rho < r + 2p - j - n$, the series $\sum'_{k \in \mathbb{Z}^n} |P_j(k)| |k|^{-r+\rho-2p}$ is summable.

Thus, $\|H(\cdot, l)\|_{\infty, \rho} \leq K F(l)$ where $K := \sum'_k |P_j(k)| |k|^{-r+\rho-2p} < \infty$. We have already seen that the series $\sum_l F(l)$ is summable, so we get the result. \square

We note that if f and g both satisfy (H1) (or (H2)), then so does $f + g$. In the following, we will use the equivalence relation

$$f \sim g \iff f - g \text{ satisfies (H1)}.$$

Lemma 2.4.13. Let f and g be two functions on $D \times (\mathbb{Z}^n)^p$ where D is an open neighborhood of 0 in \mathbb{C} , such that $f \sim g$ and such that g satisfies (H2). Then

$$\text{Res}_{s=0} \sum_{l \in (\mathbb{Z}^n)^p} f(s, l) = \sum_{l \in (\mathbb{Z}^n)^p} \text{Res}_{s=0} g(s, l).$$

Proof. Since $f \sim g$, f satisfies (H2) for a certain $\rho > 0$. Let's fix η such that $0 < \eta < \rho$ and define C_η as the circle of center 0 and radius η . We have

$$\text{Res}_{s=0} g(s, l) = \text{Res}_{s=0} f(s, l) = \frac{1}{2\pi i} \oint_{C_\eta} f(s, l) ds = \int_I u(t, l) dt.$$

where $I = [0, 2\pi]$ and $u(t, l) := \frac{1}{2\pi} \eta e^{it} f(\eta e^{it}, l)$. The fact that f satisfies (H2) entails that the series $\sum_{l \in (\mathbb{Z}^n)^p} \|f(\cdot, l)\|_{\infty, C_\eta}$ is summable. Thus, since $\|u(\cdot, l)\|_\infty = \frac{1}{2\pi} \eta \|f(\cdot, l)\|_{\infty, C_\eta}$, the series $\sum_{l \in (\mathbb{Z}^n)^p} \|u(\cdot, l)\|_\infty$ is summable, so, as a consequence, $\int_I \sum_{l \in (\mathbb{Z}^n)^p} u(t, l) dt = \sum_{l \in (\mathbb{Z}^n)^p} \int_I u(t, l) dt$ which gives the result. \square

2.4.3 Computation of residues of zeta functions

Since, we will have to compute residues of series, let us introduce the following

Definition 2.4.14.

$$\begin{aligned} \zeta(s) &:= \sum_{n=1}^{\infty} n^{-s}, \\ Z_n(s) &:= \sum'_{k \in \mathbb{Z}^n} |k|^{-s}, \\ \zeta_{p_1, \dots, p_n}(s) &:= \sum'_{k \in \mathbb{Z}^n} \frac{k_1^{p_1} \dots k_n^{p_n}}{|k|^s}, \text{ for } p_i \in \mathbb{N}, \end{aligned}$$

where $\zeta(s)$ is the Riemann zeta function (see [76] or [51]).

By the symmetry $k \rightarrow -k$, it is clear that these functions ζ_{p_1, \dots, p_n} all vanish for odd values of p_i .

Let us now compute $\zeta_{0, \dots, 0, 1_i, 0, \dots, 0, 1_j, 0, \dots, 0}(s)$ in terms of $Z_n(s)$:
Since $\zeta_{0, \dots, 0, 1_i, 0, \dots, 0, 1_j, 0, \dots, 0}(s) = A_i(s) \delta_{ij}$, exchanging the components k_i and k_j , we get

$$\zeta_{0, \dots, 0, 1_i, 0, \dots, 0, 1_j, 0, \dots, 0}(s) = \frac{\delta_{ij}}{n} Z_n(s-2).$$

Similarly,

$$\sum'_{\mathbb{Z}^n} \frac{k_1^2 k_2^2}{|k|^{s+8}} = \frac{1}{n(n-1)} Z_n(s+4) - \frac{1}{n-1} \sum'_{\mathbb{Z}^n} \frac{k_1^4}{|k|^{s+8}}$$

but it is difficult to write explicitly $\zeta_{p_1, \dots, p_n}(s)$ in terms of $Z_n(s-4)$ and other $Z_n(s-m)$ when at least four indices p_i are non zero.

When all p_i are even, $\zeta_{p_1, \dots, p_n}(s)$ is a nonzero series of fractions $\frac{P(k)}{|k|^s}$ where P is a homogeneous polynomial of degree $p_1 + \dots + p_n$. Theorem 2.4.1 now gives us the following

Proposition 2.4.15. *ζ_{p_1, \dots, p_n} has a meromorphic extension to the whole plane with a unique pole at $n + p_1 + \dots + p_n$. This pole is simple and the residue at this pole is*

$$\text{Res}_{s=n+p_1+\dots+p_n} \zeta_{p_1, \dots, p_n}(s) = 2 \frac{\Gamma(\frac{p_1+1}{2}) \dots \Gamma(\frac{p_n+1}{2})}{\Gamma(\frac{n+p_1+\dots+p_n}{2})} \quad (2.38)$$

when all p_i are even or this residue is zero otherwise.

In particular, for $n = 2$,

$$\text{Res}_{s=0} \sum'_{k \in \mathbb{Z}^2} \frac{k_i k_j}{|k|^{s+4}} = \delta_{ij} \pi, \quad (2.39)$$

and for $n = 4$,

$$\begin{aligned} \text{Res}_{s=0} \sum'_{k \in \mathbb{Z}^4} \frac{k_i k_j}{|k|^{s+6}} &= \delta_{ij} \frac{\pi^2}{2}, \\ \text{Res}_{s=0} \sum'_{k \in \mathbb{Z}^4} \frac{k_i k_j k_l k_m}{|k|^{s+8}} &= (\delta_{ij} \delta_{lm} + \delta_{il} \delta_{jm} + \delta_{im} \delta_{jl}) \frac{\pi^2}{12}. \end{aligned} \quad (2.40)$$

Proof. Equation (2.38) follows from Theorem (2.4.1)

$$\text{Res}_{s=n+p_1+\dots+p_n} \zeta_{p_1, \dots, p_n}(s) = \int_{k \in S^{n-1}} k_1^{p_1} \dots k_n^{p_n} dS(k)$$

and standard formulae (see for instance [133, VIII,1;22]). Equation (2.39) is a straightforward consequence of Equation (2.38). Equation (2.40) can be checked for the cases $i = j \neq l = m$ and $i = j = l = m$. \square

Note that $Z_n(s)$ is an Epstein zeta function associated to the quadratic form $q(x) := x_1^2 + \dots + x_n^2$, so Z_n satisfies the following functional equation

$$Z_n(s) = \pi^{s-n/2} \Gamma(n/2 - s/2) \Gamma(s/2)^{-1} Z_n(n-s).$$

Since $\pi^{s-n/2}\Gamma(n/2-s/2)\Gamma(s/2)^{-1} = 0$ for any negative even integer n and $Z_n(s)$ is meromorphic on \mathbb{C} with only one pole at $s = n$ with residue $2\pi^{n/2}\Gamma(n/2)^{-1}$ according to previous proposition, so we get $Z_n(0) = -1$. We have proved that

$$\operatorname{Res}_{s=0} Z_n(s+n) = 2\pi^{n/2}\Gamma(n/2)^{-1}, \quad (2.41)$$

$$Z_n(0) = -1. \quad (2.42)$$

2.4.4 Meromorphic continuation of a class of zeta functions

Let $n, q \in \mathbb{N}$, $q \geq 2$, and $p = (p_1, \dots, p_{q-1}) \in \mathbb{N}_0^{q-1}$.
Set $I := \{i \mid p_i \neq 0\}$ and assume that $I \neq \emptyset$ and

$$\mathcal{I} := \{\alpha = (\alpha_i)_{i \in I} \mid \forall i \in I \ \alpha_i = (\alpha_{i,1}, \dots, \alpha_{i,p_i}) \in \mathbb{N}_0^{p_i}\} = \prod_{i \in I} \mathbb{N}_0^{p_i}.$$

We will use in the sequel also the following notations:

- for $x = (x_1, \dots, x_t) \in \mathbb{R}^t$ recall that $|x|_1 = |x_1| + \dots + |x_t|$ and $|x| = \sqrt{x_1^2 + \dots + x_t^2}$;
- for all $\alpha = (\alpha_i)_{i \in I} \in \mathcal{I} = \prod_{i \in I} \mathbb{N}_0^{p_i}$,

$$|\alpha|_1 = \sum_{i \in I} |\alpha_i|_1 = \sum_{i \in I} \sum_{j=1}^{p_i} |\alpha_{i,j}| \text{ and } \binom{1/2}{\alpha} = \prod_{i \in I} \binom{1/2}{\alpha_i} = \prod_{i \in I} \prod_{j=1}^{p_i} \binom{1/2}{\alpha_{i,j}}.$$

A family of polynomials

In this paragraph we define a family of polynomials which plays an important role later.

Consider first the variables:

- for X_1, \dots, X_n we set $X = (X_1, \dots, X_n)$;
- for any $i = 1, \dots, 2q$, we consider the variables $Y_{i,1}, \dots, Y_{i,n}$ and set $Y_i := (Y_{i,1}, \dots, Y_{i,n})$ and $Y := (Y_1, \dots, Y_{2q})$;
- for $Y = (Y_1, \dots, Y_{2q})$, we set for any $1 \leq j \leq q$, $\tilde{Y}_j := Y_1 + \dots + Y_j + Y_{q+1} + \dots + Y_{q+j}$.

We define for all $\alpha = (\alpha_i)_{i \in I} \in \mathcal{I} = \prod_{i \in I} \mathbb{N}_0^{p_i}$ the polynomial

$$P_\alpha(X, Y) := \prod_{i \in I} \prod_{j=1}^{p_i} (2\langle X, \tilde{Y}_i \rangle + |\tilde{Y}_i|^2)^{\alpha_{i,j}}. \quad (2.43)$$

It is clear that $P_\alpha(X, Y) \in \mathbb{Z}[X, Y]$, $\deg_X P_\alpha \leq |\alpha|_1$ and $\deg_Y P_\alpha \leq 2|\alpha|_1$.

Let us fix a polynomial $Q \in \mathbb{R}[X_1, \dots, X_n]$ and note $d := \deg Q$. For $\alpha \in \mathcal{I}$, we want to expand $P_\alpha(X, Y)Q(X)$ in homogeneous polynomials in X and Y so defining

$$L(\alpha) := \{\beta \in \mathbb{N}_0^{(2q+1)n} : |\beta|_1 - d_\beta \leq 2|\alpha|_1 \text{ and } d_\beta \leq |\alpha|_1 + d\}$$

where $d_\beta := \sum_{i=1}^n \beta_i$, we set

$$\binom{1/2}{\alpha} P_\alpha(X, Y) Q(X) =: \sum_{\beta \in L(\alpha)} c_{\alpha, \beta} X^\beta Y^\beta$$

where $c_{\alpha, \beta} \in \mathbb{R}$, $X^\beta := X_1^{\beta_1} \dots X_n^{\beta_n}$ and $Y^\beta := Y_{1,1}^{\beta_{n+1}} \dots Y_{2q,n}^{\beta_{(2q+1)n}}$. By definition, X^β is a homogeneous polynomial of degree in X equals to d_β . We note

$$M_{\alpha, \beta}(Y) := c_{\alpha, \beta} Y^\beta.$$

Residues of a class of zeta functions

In this section we will prove the following result, used in Proposition 2.2.5 for the computation of the spectrum dimension of the noncommutative torus:

Theorem 2.4.16. (i) Let $\frac{1}{2\pi}\Theta$ be a diophantine matrix, and $\tilde{a} \in \mathcal{S}((\mathbb{Z}^n)^{2q})$. Then

$$s \mapsto f(s) := \sum_{l \in [(\mathbb{Z}^n)^q]^2} \tilde{a}_l \sum'_{k \in \mathbb{Z}^n} \prod_{i=1}^{q-1} |k + \tilde{l}_i|^{p_i} |k|^{-s} Q(k) e^{ik \cdot \Theta \sum_1^q l_j}$$

has a meromorphic continuation to the whole complex plane \mathbb{C} with at most simple possible poles at the points $s = n + d + |p|_1 - m$ where $m \in \mathbb{N}_0$.

(ii) Let $m \in \mathbb{N}_0$ and set $I(m) := \{(\alpha, \beta) \in \mathcal{I} \times \mathbb{N}_0^{(2q+1)n} : \beta \in L(\alpha) \text{ and } m = 2|\alpha|_1 - d_\beta + d\}$. Then $I(m)$ is a finite set and $s = n + d + |p|_1 - m$ is a pole of f if and only if

$$C(f, m) := \sum_{l \in Z} \tilde{a}_l \sum_{(\alpha, \beta) \in I(m)} M_{\alpha, \beta}(l) \int_{u \in S^{n-1}} u^\beta dS(u) \neq 0,$$

with $Z := \{l : \sum_1^q l_j = 0\}$ and the convention $\sum_\emptyset = 0$. In that case $s = n + d + |p|_1 - m$ is a simple pole of residue $\text{Res}_{s=n+d+|p|_1-m} f(s) = C(f, m)$.

In order to prove the theorem above we need the following

Lemma 2.4.17. For all $N \in \mathbb{N}$ we have

$$\prod_{i=1}^{q-1} |k + \tilde{l}_i|^{p_i} = \sum_{\alpha = (\alpha_i)_{i \in I} \in \prod_{i \in I} \{0, \dots, N\}^{p_i}} \binom{1/2}{\alpha} \frac{P_\alpha(k, l)}{|k|^{2|\alpha|_1 - |p|_1}} + \mathcal{O}_N(|k|^{p|_1 - (N+1)/2})$$

uniformly in $k \in \mathbb{Z}^n$ and $l \in (\mathbb{Z}^n)^{2q}$ verifying $|k| > U(l) := 36 (\sum_{i=1, i \neq q}^{2q-1} |l_i|)^4$.

Proof. For $i = 1, \dots, q-1$, we have uniformly in $k \in \mathbb{Z}^n$ and $l \in (\mathbb{Z}^n)^{2q}$ verifying $|k| > U(l)$,

$$\frac{|2\langle k, \tilde{l}_i \rangle + |\tilde{l}_i|^2|}{|k|^2} \leq \frac{\sqrt{U(l)}}{2|k|} < \frac{1}{2\sqrt{|k|}}. \quad (2.44)$$

In that case,

$$|k + \tilde{l}_i| = (|k|^2 + 2\langle k, \tilde{l}_i \rangle + |\tilde{l}_i|^2)^{1/2} = |k| \left(1 + \frac{2\langle k, \tilde{l}_i \rangle + |\tilde{l}_i|^2}{|k|^2}\right)^{1/2} = \sum_{u=0}^{\infty} \binom{1/2}{u} \frac{1}{|k|^{2u-1}} P_u^i(k, l)$$

where for all $i = 1, \dots, q-1$ and for all $u \in \mathbb{N}_0$,

$$P_u^i(k, l) := (2\langle k, \tilde{l}_i \rangle + |\tilde{l}_i|^2)^u,$$

with the convention $P_0^i(k, l) := 1$.

In particular $P_u^i(k, l) \in \mathbb{Z}[k, l]$, $\deg_k P_u^i \leq u$ and $\deg_l P_u^i \leq 2u$. Inequality (2.44) implies that for all $i = 1, \dots, q-1$ and for all $u \in \mathbb{N}$,

$$\frac{1}{|k|^{2u}} |P_u^i(k, l)| \leq (2\sqrt{|k|})^{-u}$$

uniformly in $k \in \mathbb{Z}^n$ and $l \in (\mathbb{Z}^n)^{2q}$ verifying $|k| > U(l)$.

Let $N \in \mathbb{N}$. We deduce from the previous that for any $k \in \mathbb{Z}^n$ and $l \in (\mathbb{Z}^n)^{2q}$ verifying $|k| > U(l)$ and for all $i = 1, \dots, q-1$, we have

$$\begin{aligned} |k + \tilde{l}_i| &= \sum_{u=0}^N \binom{1/2}{u} \frac{1}{|k|^{2u-1}} P_u^i(k, l) + \mathcal{O}\left(\sum_{u>N} |k| \binom{1/2}{u} (2\sqrt{|k|})^{-u}\right) \\ &= \sum_{u=0}^N \binom{1/2}{u} \frac{1}{|k|^{2u-1}} P_u^i(k, l) + \mathcal{O}_N\left(\frac{1}{|k|^{(N-1)/2}}\right). \end{aligned}$$

It follows that for any $N \in \mathbb{N}$, we have uniformly in $k \in \mathbb{Z}^n$ and $l \in (\mathbb{Z}^n)^{2q}$ verifying $|k| > U(l)$ and for all $i \in I$,

$$|k + \tilde{l}_i|^{p_i} = \sum_{\alpha_i \in \{0, \dots, N\}^{p_i}} \binom{1/2}{\alpha_i} \frac{1}{|k|^{2|\alpha_i|_1 - p_i}} P_{\alpha_i}^i(k, l) + \mathcal{O}_N\left(\frac{1}{|k|^{(N+1)/2 - p_i}}\right)$$

where $P_{\alpha_i}^i(k, l) = \prod_{j=1}^{p_i} P_{\alpha_{i,j}}^i(k, l)$ for all $\alpha_i = (\alpha_{i,1}, \dots, \alpha_{i,p_i}) \in \{0, \dots, N\}^{p_i}$ and

$$\prod_{i \in I} |k + \tilde{l}_i|^{p_i} = \sum_{\alpha = (\alpha_i) \in \prod_{i \in I} \{0, \dots, N\}^{p_i}} \binom{1/2}{\alpha} \frac{1}{|k|^{2|\alpha|_1 - |p|_1}} P_{\alpha}(k, l) + \mathcal{O}_N\left(\frac{1}{|k|^{(N+1)/2 - |p|_1}}\right)$$

where $P_{\alpha}(k, l) = \prod_{i \in I} P_{\alpha_i}^i(k, l) = \prod_{i \in I} \prod_{j=1}^{p_i} P_{\alpha_{i,j}}^i(k, l)$. \square

Proof of Theorem 2.4.16.

(i) All $n, q, p = (p_1, \dots, p_{q-1})$ and $\tilde{a} \in \mathcal{S}((\mathbb{Z}^n)^{2q})$ are fixed as above and we define formally for any $l \in (\mathbb{Z}^n)^{2q}$

$$F(l, s) := \sum'_{k \in \mathbb{Z}^n} \prod_{i=1}^{q-1} |k + \tilde{l}_i|^{p_i} Q(k) e^{ik \cdot \Theta \sum_1^q l_j} |k|^{-s}. \quad (2.45)$$

Thus, still formally,

$$f(s) := \sum_{l \in (\mathbb{Z}^n)^{2q}} \tilde{a}_l F(l, s). \quad (2.46)$$

It is clear that $F(l, s)$ converges absolutely in the half plane $\{\sigma = \Re(s) > n + d + |p|_1\}$ where $d = \deg Q$.

Let $N \in \mathbb{N}$. Lemma 2.4.17 implies that for any $l \in (\mathbb{Z}^n)^{2q}$ and for $s \in \mathbb{C}$ such that $\sigma > n + |p|_1 + d$,

$$\begin{aligned} F(l, s) &= \sum'_{|k| \leq U(l)} \prod_{i=1}^{q-1} |k + \tilde{l}_i|^{p_i} Q(k) e^{ik \cdot \Theta \sum_1^q l_j} |k|^{-s} \\ &\quad + \sum_{\alpha = (\alpha_i)_{i \in I} \in \prod_{i \in I} \{0, \dots, N\}^{p_i}} \binom{1/2}{\alpha} \sum_{|k| > U(l)} \frac{1}{|k|^{s+2|\alpha|_1 - |p|_1}} P_{\alpha}(k, l) Q(k) e^{ik \cdot \Theta \sum_1^q l_j} + G_N(l, s). \end{aligned}$$

where $s \mapsto G_N(l, s)$ is a holomorphic function in the half-plane $D_N := \{\sigma > n + d + |p|_1 - \frac{N+1}{2}\}$ and verifies in it the bound $G_N(l, s) \ll_{N, \sigma} 1$ uniformly in l .

It follows that

$$F(l, s) = \sum_{\alpha=(\alpha_i)_{i \in I} \in \prod_{i \in I} \{0, \dots, N\}^{p_i}} H_\alpha(l, s) + R_N(l, s), \quad (2.47)$$

where

$$\begin{aligned} H_\alpha(l, s) &:= \sum'_{k \in \mathbb{Z}^n} \binom{1/2}{\alpha} \frac{1}{|k|^{s+2|\alpha|_1-|p|_1}} P_\alpha(k, l) Q(k) e^{ik \cdot \Theta \sum_1^q l_j}, \\ R_N(l, s) &:= \sum'_{|k| \leq U(l)} \prod_{i=1}^{q-1} |k + \tilde{l}_i|^{p_i} Q(k) e^{ik \cdot \Theta \sum_1^q l_j} |k|^{-s} \\ &\quad - \sum'_{|k| \leq U(l)} \sum_{\alpha=(\alpha_i)_{i \in I} \in \prod_{i \in I} \{0, \dots, N\}^{p_i}} \binom{1/2}{\alpha} \frac{P_\alpha(k, l)}{|k|^{s+2|\alpha|_1-|p|_1}} Q(k) e^{ik \cdot \Theta \sum_1^q l_j} + G_N(l, s). \end{aligned}$$

In particular there exists $A(N) > 0$ such that $s \mapsto R_N(l, s)$ extends holomorphically to the half-plane D_N and verifies in it the bound $R_N(l, s) \ll_{N, \sigma} 1 + |l|^{A(N)}$ uniformly in l .

Let us note formally

$$h_\alpha(s) := \sum_l \tilde{a}_l H_\alpha(l, s).$$

Equation (2.47) and $R_N(l, s) \ll_{N, \sigma} 1 + |l|^{A(N)}$ imply that

$$f(s) \sim_N \sum_{\alpha=(\alpha_i)_{i \in I} \in \prod_{i \in I} \{0, \dots, N\}^{p_i}} h_\alpha(s), \quad (2.48)$$

where \sim_N means modulo a holomorphic function in D_N .

Recall the decomposition $\binom{1/2}{\alpha} P_\alpha(k, l) Q(k) = \sum_{\beta \in L(\alpha)} M_{\alpha, \beta}(l) k^\beta$ and we decompose similarly $h_\alpha(s) = \sum_{\beta \in L(\alpha)} h_{\alpha, \beta}(s)$. Theorem 2.4.4 now implies that for all $\alpha = (\alpha_i)_{i \in I} \in \prod_{i \in I} \{0, \dots, N\}^{p_i}$ and $\beta \in L(\alpha)$,

- the map $s \mapsto h_{\alpha, \beta}(s)$ has a meromorphic continuation to the whole complex plane \mathbb{C} with only one simple possible pole at $s = n + |p|_1 - 2|\alpha|_1 + d_\beta$,
- the residue at this point is equal to

$$\operatorname{Res}_{s=n+|p|_1-2|\alpha|_1+d_\beta} h_{\alpha, \beta}(s) = \sum_{l \in \mathcal{Z}} \tilde{a}_l M_{\alpha, \beta}(l) \int_{u \in S^{n-1}} u^\beta dS(u) \quad (2.49)$$

where $\mathcal{Z} := \{l \in (\mathbb{Z})^{2q} : \sum_1^q l_j = 0\}$. If the right hand side is zero, $h_{\alpha, \beta}(s)$ is holomorphic on \mathbb{C} .

By (2.48), we deduce therefore that $f(s)$ has a meromorphic continuation on the halfplane D_N , with only simple possible poles in the set $\{n + |p|_1 + k : -2N|p|_1 \leq k \leq d\}$. Taking now $N \rightarrow \infty$ yields the result.

(ii) Let $m \in \mathbb{N}_0$ and set $I(m) := \{(\alpha, \beta) \in \mathcal{I} \times \mathbb{N}_0^{(2q+1)n} : \beta \in L(\alpha) \text{ and } m = 2|\alpha|_1 - d_\beta + d\}$. If $(\alpha, \beta) \in I(m)$, then $|\alpha|_1 \leq m$ and $|\beta|_1 \leq 3m + d$, so $I(m)$ is finite.

With a chosen N such that $2N|p|_1 + d > m$, we get by (2.48) and (2.49)

$$\operatorname{Res}_{s=n+d+|p|_1-m} f(s) = \sum_{l \in \mathcal{Z}} \tilde{a}_l \sum_{(\alpha, \beta) \in I(m)} M_{\alpha, \beta}(l) \int_{u \in S^{n-1}} u^\beta dS(u) = C(f, m)$$

with the convention $\sum_\emptyset = 0$. Thus, $n + d + |p|_1 - m$ is a pole of f if and only if $C(f, m) \neq 0$. \square

Chapitre 3

Spectral action on $SU_q(2)$

3.1 Introduction

The quantum group $SU_q(2)$ has already a rather long history of studies [92] being one of the finest examples of quantum deformation. This includes an approach via the noncommutative notion of spectral triple introduced by Connes [28, 37] and various notions of Dirac operators were introduced in [8, 18, 20, 31, 67]. Finally, a real spectral triple, which was exhibited in [48], is invariant by left and right action of $\mathcal{U}_q(su(2))$ and satisfies almost all postulated axioms of triples except the commutant and first-order properties. These, however, remain valid only up to infinitesimal of arbitrary high order. The last presentation generalizes in a straightforward way all geometric construction details of the spinorial spectral triple for the classical three-sphere. In particular, both the equivariant representation and the symmetries have a $q \rightarrow 1$ proper classical limit.

The goal of this chapter is to obtain the spectral action on $SU_q(2)$ which is a spectral triple with an invertible Dirac operator, with the control of the differential calculus generated by the Dirac operator arising as the main difficulty. This issue of computing the spectral action was addressed in the epilogue of [137]. In the case of $SU_q(2)$, we have $Sd^+ = Sd = \{1, 2, 3\}$, so

$$\mathcal{S}(\mathcal{D}_\mathbb{A}, \Phi, \Lambda) = \sum_{1 \leq k \leq 3} \Phi_k \Lambda^k \oint |D_\mathbb{A}|^{-k} + \Phi(0) \zeta_{D_\mathbb{A}}(0). \quad (3.1)$$

Note that in the case of $SU_q(2)$ there are no terms in Λ^{-k} , $k > 0$ because the dimension spectrum is bounded below by 1.

To proceed with the computation of (3.1), we introduce two presentations of one-forms. The main ingredient is $F = \text{sign } (\mathcal{D})$ which appears to be a one-form up to $OP^{-\infty}$.

In section 2, we discuss the spectral action of an arbitrary 3-dimensional spectral triple using cocycles.

In sections 3 and 4 we recall the main results on $SU_q(2)$ of [48] and show that the full spectral action with reality operator given by (1.6) is completely determined by the terms

$$\oint A^q |\mathcal{D}|^{-p}, \quad 1 \leq q \leq p \leq 3.$$

where A is a linear combination of terms of the form $a[|\mathcal{D}|, b]$ with $a, b \in \mathcal{A}$.

In section 5, we establish a differential calculus up to some ideal in pseudodifferential operators and apply these results to the precise computation of previous noncommutative integrals.

Section 6 is devoted to explicit examples, while in next section are given different comparisons with the commutative case of the 3-sphere corresponding to $SU(2)$.

3.2 Spectral action in 3-dimension

3.2.1 Tadpole and cocycles

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple of dimension 3. We refer to [22, 37] for the definition of the OP^α spaces and the algebra of pseudodifferential operators $\Psi(\mathcal{A})$ on a spectral triple.

For $n \in \mathbb{N}^*$ and $a_i \in \mathcal{A}$, define

$$\phi_n(a_0, \dots, a_n) := \oint a_0[\mathcal{D}, a_1]D^{-1} \dots [\mathcal{D}, a_n]D^{-1}.$$

We also use notational integrals on the universal n -forms $\Omega_u^n(\mathcal{A})$ defined by

$$\int_{\phi_n} a_0 da_1 \dots da_n := \phi_n(a_0, a_1, \dots, a_n).$$

and the reordering fact that $(da_0)a_1 = d(a_0a_1) - a_0da_1$.

We use the $b - B$ bicomplex defined in [28]: b is the Hochschild coboundary map (and b' is truncated one) defined on n -cochains ϕ by

$$\begin{aligned} b\phi(a_0, \dots, a_{n+1}) &:= b'\phi(a_0, \dots, a_{n+1}) + (-1)^{n+1}\phi(a_{n+1}a_0, a_1, \dots, a_n), \\ b'\phi(a_0, \dots, a_{n+1}) &:= \sum_{j=0}^n (-1)^j \phi(a_0, \dots, a_j a_{j+1}, \dots, a_{n+1}). \end{aligned}$$

Recall that B_0 is defined on the normalized cochains ϕ_n by

$$B_0\phi_n(a_0, a_1, \dots, a_{n-1}) := \phi_n(1, a_0, \dots, a_{n-1}), \text{ thus } \int_{\phi_n} d\omega = \int_{B_0\phi_n} \omega \text{ for } \omega \in \Omega_u^{n-1}(\mathcal{A}).$$

Then $B := NB_0$, where $N := 1 + \lambda + \dots \lambda^n$ is the cyclic skewsymmetrizer on the n -cochains and λ is the cyclic permutation $\lambda\phi(a_0, \dots, a_n) := (-1)^n\phi(a_n, a_0, \dots, a_{n-1})$.

We will also encounter the cyclic 1-cochain $N\phi_1$:

$$N\phi_1(a_0, a_1) := \phi_1(a_0, a_1) - \phi_1(a_1, a_0) \text{ and } \int_{N\phi_1} a_0 da_1 := N\phi_1(a_0, a_1).$$

Remark 3.2.1. Assume the integrand of f is in OP^{-3} . Since $[D^{-1}, a] = -D^{-1}[D, a]D^{-1} \in OP^{-2}$, this commutator introduces an integrand in OP^{-4} so has a vanishing integral: under the integral, we can commute D^{-1} with all $a \in \mathcal{A}$, but not with one-forms. Note also that since $P_0 \in OP^{-\infty}$, any integrand containing P_0 has a vanishing integral.

Lemma 3.2.2. *We have*

- (i) $b\phi_1 = -\phi_2.$
- (ii) $b\phi_2 = 0.$
- (iii) $b\phi_3 = 0.$
- (iv) $B\phi_1 = 0.$
- (v) $B_0\phi_2 = -(1 - \lambda)\phi_1.$
- (vi) $bB_0\phi_2 = 2\phi_2 + B_0\phi_3.$
- (vii) $B\phi_2 = 0.$
- (viii) $B_0\phi_3 = Nb'\phi_1.$
- (ix) $B\phi_3 = 3B_0\phi_3.$

Proof. (i)

$$\begin{aligned} b\phi_1(a_0, a_1, a_2) &= \oint a_0 a_1 [\mathcal{D}, a_2] D^{-1} - \oint a_0 (a_1 [\mathcal{D}, a_2] + [\mathcal{D}, a_1] a_2) D^{-1} + \oint a_2 a_0 [\mathcal{D}, a_1] D^{-1} \\ &= \oint a_0 [\mathcal{D}, a_1] (D^{-1} a_2 - a_2 D^{-1}) = - \oint a_0 [\mathcal{D}, a_1] D^{-1} [\mathcal{D}, a_2] D^{-1} \\ &= -\phi_2(a_0, a_1, a_2) \end{aligned}$$

where we have used the trace property of the noncommutative integral.

(ii) $b\phi_2(a_0, a_1, a_2, a_3)$

$$\begin{aligned} &= \oint a_0 a_1 [\mathcal{D}, a_2] D^{-1} [\mathcal{D}, a_3] D^{-1} - \oint a_0 (a_1 [\mathcal{D}, a_2] + [\mathcal{D}, a_1] a_2) D^{-1} [\mathcal{D}, a_3] D^{-1} \\ &\quad + \oint a_0 [\mathcal{D}, a_1] D^{-1} (a_2 [\mathcal{D}, a_3] + [\mathcal{D}, a_2] a_3) D^{-1} - \oint a_3 a_0 [\mathcal{D}, a_1] D^{-1} [\mathcal{D}, a_2] D^{-1} \\ &= \oint a_0 [\mathcal{D}, a_1] (D^{-1} a_2 - a_2 D^{-1}) [\mathcal{D}, a_3] D^{-1} + \oint a_0 [\mathcal{D}, a_1] D^{-1} [\mathcal{D}, a_2] (a_3 D^{-1} - D^{-1} a_3) \\ &= - \oint a_0 [\mathcal{D}, a_1] D^{-1} [\mathcal{D}, a_2] D^{-1} [\mathcal{D}, a_3] D^{-1} + \oint a_0 [\mathcal{D}, a_1] D^{-1} [\mathcal{D}, a_2] D^{-1} [\mathcal{D}, a_3] D^{-1} \\ &= 0. \end{aligned}$$

(iii) Using Remark 3.2.1, we get $\phi_3(a_0, a_1, a_2, a_3) = \oint a_0 [\mathcal{D}, a_1] [\mathcal{D}, a_2] [\mathcal{D}, a_3] |\mathcal{D}|^{-3}$, so similar computations as for ϕ_2 gives $b\phi_3 = 0$.

(iv) $B_0\phi_1(a_0) = \oint [\mathcal{D}, a_0] D^{-1} = \oint (\mathcal{D} a_0 D^{-1} - a_0) = 0.$

$$\begin{aligned} (v) \quad B_0\phi_2(a_0, a_1) &= \oint [\mathcal{D}, a_0] D^{-1} [\mathcal{D}, a_1] D^{-1} = \oint a_0 D^{-1} [\mathcal{D}, a_1] - \oint a_0 [\mathcal{D}, a_1] D^{-1} \\ &= \oint a_0 a_1 - \oint a_0 D^{-1} a_1 \mathcal{D} - \oint a_0 [\mathcal{D}, a_1] D^{-1} \\ &= - \oint a_1 [\mathcal{D}, a_0] D^{-1} - \oint a_0 [\mathcal{D}, a_1] D^{-1} = -\phi_1(a_1, a_0) - \phi_1(a_0, a_1). \end{aligned}$$

(vi) Since $-b\lambda\phi_1(a_0, a_1, a_2) = \phi_1(a_2, a_0 a_1) - \phi_1(a_1 a_2, a_0) + \phi_1(a_1, a_2 a_0)$, one obtains that

$$-b\lambda\phi_1(a_0, a_1, a_2) = \oint a_0 a_1 D^{-1} a_2 \mathcal{D} + a_0 D^{-1} a_1 \mathcal{D} a_2 - a_0 D^{-1} a_1 a_2 \mathcal{D} - a_0 a_1 a_2.$$

So by direct expansion, this is equal to $-\oint a_0 D^{-1}[\mathcal{D}, a_1] D^{-1}[\mathcal{D}, a_2]$ which means that

$$\begin{aligned} -b\lambda\phi_1(a_0, a_1, a_2) &= \oint [D^{-1}, a_0][\mathcal{D}, a_1] D^{-1}[\mathcal{D}, a_2] - a_0[\mathcal{D}, a_1] D^{-1}[\mathcal{D}, a_2] D^{-1} \\ &= -B_0\phi_3(a_0, a_1, a_2) - \phi_2(a_0, a_1, a_2). \end{aligned}$$

Now the result follows from (i), (v).

(vii) $B\phi_2 = NB_0\phi_2 = -N(1-\lambda)\phi_1 = 0$ since $N(1-\lambda) = 0$.

$$\begin{aligned} (viii) \quad B_0\phi_3(a_0, a_1, a_2) &= \oint [\mathcal{D}, a_0] D^{-1}[\mathcal{D}, a_1] D^{-1}[\mathcal{D}, a_2] D^{-1} \\ &= \oint a_0 D^{-1}[\mathcal{D}, a_1] D^{-1}[\mathcal{D}, a_2] - \oint a_0[\mathcal{D}, a_1] D^{-1}[\mathcal{D}, a_2] D^{-1} \\ &= \oint a_0 a_1 D^{-1}[\mathcal{D}, a_2] - \oint a_0 D^{-1} a_1[\mathcal{D}, a_2] - \oint a_0[\mathcal{D}, a_1] D^{-1}[\mathcal{D}, a_2] D^{-1} \\ &= \oint a_0 a_1 a_2 - \oint a_0 a_1 D^{-1} a_2 \mathcal{D} - \oint a_0 D^{-1} a_1 \mathcal{D} a_2 + \oint a_0 D^{-1} a_1 a_2 \mathcal{D} \\ &\quad - \oint a_0[\mathcal{D}, a_1] D^{-1}[\mathcal{D}, a_2] D^{-1} \\ &= \oint a_0 a_1 a_2 - a_2 \mathcal{D} a_1 a_0 D^{-1} + a_1 a_2 \mathcal{D} a_0 D^{-1} + a_2 \mathcal{D} a_0 a_1 D^{-1} \\ &\quad - (a_0 \mathcal{D} a_1 a_2 D^{-1} - a_0 \mathcal{D} a_1 D^{-1} - a_0 a_1 \mathcal{D} a_2 D^{-1} + a_0 a_1 a_2). \end{aligned}$$

Expanding $(id + \lambda + \lambda^2)b'\phi_1(a_0, a_1, a_2)$, we recover previous expression.

(ix) Consequence of (viii). □

3.2.2 Scale-invariant term of the spectral action

We know from [22] that the scale-invariant term of the action can be written as

$$\zeta_{D_A}(0) - \zeta_D(0) = -\oint AD^{-1} + \frac{1}{2}\oint AD^{-1}AD^{-1} - \frac{1}{3}\oint AD^{-1}AD^{-1}AD^{-1}. \quad (3.2)$$

In fact, this action can be expressed in dimension 3 as contributions corresponding to tadpoles and the Yang–Mills and Chern–Simons actions in dimension 4:

Proposition 3.2.3. *For any one-form A ,*

$$\zeta_{D_A}(0) - \zeta_D(0) = -\frac{1}{2}\int_{N\phi_1} A + \frac{1}{2}\int_{\phi_2} (dA + A^2) - \frac{1}{2}\int_{\phi_3} (AdA + \frac{2}{3}A^3). \quad (3.3)$$

To prove this, we calculate now each terms of the action.

Lemma 3.2.4. *For any one-form A , we have*

- (i) $\int_{\phi_2} dA = \int_{B_0\phi_2} A = -\int_{\phi_1} A + \int_{\lambda\phi_1} A$.
- (ii) $\oint AD^{-1} = \int_{\phi_1} A = \frac{1}{2}\int_{N\phi_1} A - \frac{1}{2}\int_{\phi_2} dA$.
- (iii) $\oint AD^{-1}AD^{-1} = -\int_{\phi_3} AdA + \int_{\phi_2} A^2$.
- (iv) $\oint AD^{-1}AD^{-1}AD^{-1} = \int_{\phi_3} A^3$.

Proof. (i) and (ii) follow directly from Lemma 3.2.2 (v).

(iii) With the shorthand $A = a_i db_i$ (summation on i)

$$\begin{aligned} \oint AD^{-1}AD^{-1} &= \oint a_0[\mathcal{D}, b_0]D^{-1}a_1[\mathcal{D}, b_1]D^{-1} \\ &= - \int_{\phi_3} AdA + \oint a_0[\mathcal{D}, b_0]a_1b_1D^{-1} - \oint a_0[\mathcal{D}, b_0]a_1D^{-1}b_1. \end{aligned}$$

We calculate further the remaining terms

$$\begin{aligned} \oint a_0[\mathcal{D}, b_0]a_1b_1D^{-1} - \oint a_0[\mathcal{D}, b_0]a_1D^{-1}b_1 &= \oint a_0\mathcal{D}b_0a_1b_1D^{-1} - \oint a_0b_0\mathcal{D}a_1b_1D^{-1} \\ &\quad - \oint a_0\mathcal{D}b_0a_1D^{-1}b_1 + \oint a_0b_0\mathcal{D}a_1D^{-1}b_1, \end{aligned}$$

which are compared with $\int_{\phi_2} A^2 = \int_{\phi_2} a_0(db_0)a_1db_1 = \int_{\phi_2} a_0d(b_0a_1)db_1 - a_0b_0da_1db_1$:

$$\begin{aligned} \int_{\phi_2} A^2 &= \oint a_0[\mathcal{D}, b_0a_1]D^{-1}[\mathcal{D}, b_1]D^{-1} - \oint a_0b_0[\mathcal{D}, a_1]D^{-1}[\mathcal{D}, b_1]D^{-1} \\ &= \oint a_0\mathcal{D}b_0a_1b_1D^{-1} - \oint a_0\mathcal{D}b_0a_1D^{-1}b_1 - \oint a_0b_0a_1\mathcal{D}b_1D^{-1} + \oint a_0b_0a_1b_1 \\ &\quad - \oint a_0b_0\mathcal{D}a_1b_1D^{-1} + \oint a_0b_0\mathcal{D}a_1D^{-1}b_1 + \oint a_0b_0a_1\mathcal{D}b_1D^{-1} - \oint a_0b_0a_1b_1 \\ &= \oint a_0\mathcal{D}b_0a_1b_1D^{-1} - \oint b_1a_0\mathcal{D}b_0a_1D^{-1} - \oint a_0b_0\mathcal{D}a_1b_1D^{-1} + \oint b_1a_0b_0\mathcal{D}a_1D^{-1}. \end{aligned}$$

(iv) Note that

$$\begin{aligned} \int_{\phi_3} A^3 &= \int_{\phi_3} a_0(db_0)a_1(db_1)a_2db_2 = \int_{\phi_3} a_0d(b_0a_1)d(b_1a_2)db_2 - a_0b_0da_1d(b_1a_2)db_2 \\ &\quad - a_0d(b_0a_1b_1)d(a_2db_2 + a_0b_0d(a_1b_1)da_2db_2 \\ &= \oint a_0[\mathcal{D}, b_0a_1]D^{-1}[\mathcal{D}, b_1a_2]D^{-1}[\mathcal{D}, b_2]D^{-1} - a_0b_0[\mathcal{D}, a_1]D^{-1}[\mathcal{D}, b_1a_2]D^{-1}[\mathcal{D}, b_2]D^{-1} \\ &\quad - a_0[\mathcal{D}, b_0a_1b_1]D^{-1}[\mathcal{D}, a_2]D^{-1}[\mathcal{D}, b_2]D^{-1} + a_0b_0[\mathcal{D}, a_1b_1]D^{-1}[\mathcal{D}, a_2]D^{-1}[\mathcal{D}, b_2]D^{-1}. \end{aligned}$$

Summing up the first two terms and the last two ones gives

$$\int_{\phi_3} A^3 = \oint a_0[\mathcal{D}, b_0]a_1D^{-1}[\mathcal{D}, b_1a_2]D^{-1}[\mathcal{D}, b_2]D^{-1} - a_0[\mathcal{D}, b_0]a_1b_1D^{-1}[\mathcal{D}, a_2]D^{-1}[\mathcal{D}, b_2]D^{-1}.$$

Using Remark 3.2.1, we can commute under the integral D^{-1} with all $a \in \mathcal{A}$ and similarly

$$\oint AD^{-1}AD^{-1}AD^{-1} = \oint a_0[\mathcal{D}, b_0]a_1D^{-1}[\mathcal{D}, b_1]a_2D^{-1}[\mathcal{D}, b_2]D^{-1}$$

which proves (iv). \square

We deduce Proposition 3.2.3 from (3.2) using the previous lemma.

3.3 The $SU_q(2)$ triple

3.3.1 The spectral triple

We briefly recall the main facts of the real spectral triple $(\mathcal{A}(SU_q(2)), \mathcal{H}, \mathcal{D})$ introduced in [48], see also [18, 19, 31].

The algebra:

Let $\mathcal{A} := \mathcal{A}(SU_q(2))$ be the $*$ -algebra generated polynomially by a and b , subject to the following commutation rules with $0 < q < 1$:

$$ba = qab, \quad b^*a = qab^*, \quad bb^* = b^*b, \quad a^*a + q^2b^*b = 1, \quad aa^* + bb^* = 1. \quad (3.4)$$

We recall the following lemma from [153, Lemma A2.1]:

Lemma 3.3.1. *For any representation π of \mathcal{A} ,*

$$\begin{aligned} \text{Spect}(\pi(bb^*)) &= \{0, q^{2k} \mid k \in \mathbb{N}\} \text{ or } \pi(b) = 0, \\ \text{Spect}(\pi(aa^*)) &= \{1, 1 - q^{2k} \mid k \in \mathbb{N}\} \text{ or } \pi(b) = 0 \text{ and } \pi(a) \text{ is a unitary.} \end{aligned}$$

This result is interesting since it shows the appearance of discreteness for $0 \leq q < 1$ while for $q = 1$, $SU_q(2) = SU(2) \simeq \mathbb{S}^3$ and the spectrum of the commuting operator $\pi(aa^*)$ and $\pi(bb^*)$ are equal to $[0, 1]$. Moreover, all foregoing results on noncommutative integrals will involve q^2 and not q .

Any element of \mathcal{A} can be uniquely decomposed as a linear combination of terms of the form $a^\alpha b^\beta b^{*\gamma}$ where $\alpha \in \mathbb{Z}$, $\beta, \gamma \in \mathbb{N}$, with the convention

$$a^{-|\alpha|} := a^{*|\alpha|}.$$

The spinorial Hilbert space:

$\mathcal{H} = \mathcal{H}^\uparrow \oplus \mathcal{H}^\downarrow$ has an orthonormal basis consisting of vectors $|j\mu n^\uparrow\rangle$ with $j = 0, \frac{1}{2}, 1, \dots$, $\mu = -j, \dots, j$ and $n = -j^+, \dots, j^+$, together with $|j\mu n^\downarrow\rangle$ for $j = \frac{1}{2}, 1, \dots$, $\mu = -j, \dots, j$ and $n = -j^-, \dots, j^-$ (here $x^\pm := x \pm \frac{1}{2}$).

It is convenient to use a vector notation, setting:

$$|j\mu n\rangle\rangle := \begin{pmatrix} |j\mu n^\uparrow\rangle \\ |j\mu n^\downarrow\rangle \end{pmatrix} \quad (3.5)$$

and with the convention that the lower component is zero when $n = \pm(j + \frac{1}{2})$ or $j = 0$.

The representation π and its approximate $\underline{\pi}$:

It is known that representation theory of $SU_q(2)$ is similar to that of $SU(2)$ [153]. The representation π given in [48] is:

$$\begin{aligned} \pi(a) |j\mu n\rangle\rangle &:= \alpha_{j\mu n}^+ |j^+ \mu^+ n^+\rangle\rangle + \alpha_{j\mu n}^- |j^- \mu^+ n^+\rangle\rangle, \\ \pi(b) |j\mu n\rangle\rangle &:= \beta_{j\mu n}^+ |j^+ \mu^+ n^-\rangle\rangle + \beta_{j\mu n}^- |j^- \mu^+ n^-\rangle\rangle, \\ \pi(a^*) |j\mu n\rangle\rangle &:= \tilde{\alpha}_{j\mu n}^+ |j^+ \mu^- n^-\rangle\rangle + \tilde{\alpha}_{j\mu n}^- |j^- \mu^- n^-\rangle\rangle, \\ \pi(b^*) |j\mu n\rangle\rangle &:= \tilde{\beta}_{j\mu n}^+ |j^+ \mu^- n^+\rangle\rangle + \tilde{\beta}_{j\mu n}^- |j^- \mu^- n^+\rangle\rangle \end{aligned} \quad (3.6)$$

where

$$\begin{aligned}\alpha_{j\mu n}^+ &:= \sqrt{q^{\mu+n-1/2}[j+\mu+1]} \begin{pmatrix} q^{-j-1/2} \frac{\sqrt{[j+n+3/2]}}{[2j+2]} & 0 \\ q^{1/2} \frac{\sqrt{[j-n+1/2]}}{[2j+1][2j+2]} & q^{-j} \frac{\sqrt{[j+n+1/2]}}{[2j+1]} \end{pmatrix}, \\ \alpha_{j\mu n}^- &:= \sqrt{q^{\mu+n+1/2}[j-\mu]} \begin{pmatrix} q^{j+1} \frac{\sqrt{[j-n+1/2]}}{[2j+1]} & -q^{1/2} \frac{\sqrt{[j+n+1/2]}}{[2j][2j+1]} \\ 0 & q^{j+1/2} \frac{\sqrt{[j-n-1/2]}}{[2j]} \end{pmatrix}, \\ \beta_{j\mu n}^+ &:= \sqrt{q^{\mu+n-1/2}[j+\mu+1]} \begin{pmatrix} \frac{\sqrt{[j-n+3/2]}}{[2j+2]} & 0 \\ -q^{-j-1} \frac{\sqrt{[j+n+1/2]}}{[2j+1][2j+2]} & q^{-1/2} \frac{\sqrt{[j-n+1/2]}}{[2j+1]} \end{pmatrix}, \\ \beta_{j\mu n}^- &:= \sqrt{q^{\mu+n-1/2}[j-\mu]} \begin{pmatrix} -q^{-1/2} \frac{\sqrt{[j+n+1/2]}}{[2j+1]} & -q^j \frac{\sqrt{[j-n+1/2]}}{[2j][2j+1]} \\ 0 & -\frac{\sqrt{[j+n-1/2]}}{[2j]} \end{pmatrix}\end{aligned}$$

with $\tilde{\alpha}_{j\mu n}^\pm := (\alpha_{j\pm\mu-n}^\mp)^*$, $\tilde{\beta}_{j\mu n}^\pm := (\beta_{j\pm\mu-n}^\mp)^*$ and with the q -number of $\alpha \in \mathbb{R}$ be defined as

$$[\alpha] := \frac{q^\alpha - q^{-\alpha}}{q - q^{-1}}.$$

For the purpose of this chapter it is sufficient to use the approximate spinorial $*$ -representation $\underline{\pi}$ of $SU_q(2)$ presented in [48, 138] instead of the full spinorial one π .

This approximate representation is

$$\underline{\pi}(a) := a_+ + a_-, \quad \underline{\pi}(b) := b_+ + b_-$$

with the following definitions, where $q_n := \sqrt{1 - q^{2n}}$:

$$\begin{aligned}a_+ |j\mu n\rangle &:= q_{j++\mu+} \begin{pmatrix} q_{j++n+1} & 0 \\ 0 & q_{j++n} \end{pmatrix} |j^+ \mu^+ n^+\rangle, \\ a_- |j\mu n\rangle &:= q^{2j+\mu+n+\frac{1}{2}} \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} |j^- \mu^+ n^+\rangle, \\ b_+ |j\mu n\rangle &:= q^{j+n-\frac{1}{2}} q_{j++\mu+} \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} |j^+ \mu^+ n^-\rangle, \\ b_- |j\mu n\rangle &:= -q^{j+\mu} \begin{pmatrix} q_{j++n} & 0 \\ 0 & q_{j--n} \end{pmatrix} |j^- \mu^+ n^-\rangle.\end{aligned}\tag{3.7}$$

All disregarded terms are trace-class and do not influence residue calculations. More precisely, $\pi(x) - \underline{\pi}(x) \in \mathcal{K}_q$ where \mathcal{K}_q is the principal ideal generated by the operator

$$J_q |j\mu n\rangle := q^j |j\mu n\rangle.\tag{3.8}$$

Actually, \mathcal{K}_q is independent of q and is contained in all ideals of operators such that $\mu_n = o(n^{-\alpha})$ (infinitesimal of order α) for any $\alpha > 0$, and $\mathcal{K}_q \subset OP^{-\infty}$.

We define the alternative orthonormal basis $v_{m,l}^{j\uparrow}$ and $v_{m,l}^{j\downarrow}$ and the vector notation

$$v_{m,l}^j := \begin{pmatrix} v_{m,l}^{j\uparrow} \\ v_{m,l}^{j\downarrow} \end{pmatrix} \text{ where } v_{m,l}^{j\uparrow} := |j, m-j, l-j^+, \uparrow\rangle, \quad v_{m,l}^{j\downarrow} := |j, m-j, l-j^-, \downarrow\rangle.$$

Here $j \in \frac{1}{2}\mathbb{N}$, $0 \leq m \leq 2j$, $0 \leq l \leq 2j+1$ and $v_{m,l}^{l,j}$ is zero whenever $j = 0$ or $l = 2j$ or $2j+1$. The interest is that now, the operators a_{\pm} and b_{\pm} assume simpler form:

$$\begin{aligned} a_+ v_{m,l}^j &= q_{m+1} q_{l+1} v_{m+1,l+1}^{j+}, & a_- v_{m,l}^j &= q^{m+l+1} v_{m,l}^{j-}, \\ b_+ v_{m,l}^j &= q^l q_{m+1} v_{m+1,l}^{j+}, & b_- v_{m,l}^j &= -q^m q_l v_{m,l-1}^{j-}. \end{aligned} \quad (3.9)$$

Thus

$$\begin{aligned} a_+^* v_{m,l}^j &= q_m q_l v_{m-1,l-1}^{j-}, & a_-^* v_{m,l}^j &= q^{m+l+1} v_{m,l}^{j+}, \\ b_+^* v_{m,l}^j &= q^l q_m v_{m-1,l}^{j-}, & b_-^* v_{m,l}^j &= -q^m q_{l+1} v_{m,l+1}^{j+}. \end{aligned} \quad (3.10)$$

Moreover, we have

$$\begin{aligned} a_- a_+ &= q^2 a_+ a_-, & b_- b_+ &= q^2 b_+ b_-, & b_+ a_+ &= q a_+ b_+, & b_- a_- &= q a_- b_-, \\ a_-^* a_+ &= q^2 a_+ a_-^*, & a_-^* a_- &= a_- a_-^*, & a_-^* b_+ &= q b_+ a_-^*, & a_-^* b_- &= q b_- a_-^*, \\ a_+^* a_- &= q^2 a_- a_+^*, & b_-^* b_+ &= b_+ b_-^*, & b_-^* a_+ &= q a_+ b_-^*, & a_- b_+ &= q b_+ a_- . \end{aligned} \quad (3.11)$$

Note for instance that

$$\begin{aligned} a_+ a_+^* v_{m,l}^j &= q_m^2 q_l^2 v_{m,l}^j, & a_+^* a_+ v_{m,l}^j &= q_{m+1}^2 q_{l+1}^2 v_{m,l}^j, \\ b_+ b_+^* v_{m,l}^j &= q^{2l} q_m^2 v_{m,l}^j, & b_+^* b_+ v_{m,l}^j &= q^{2l} q_{m+1}^2 v_{m,l}^j, \end{aligned}$$

so applied to $v_{m,l}^j$, we get the first relation (and similarly for the others)

$$a_+^* a_+ - q^2 a_+ a_+^* + q^2 (b_+^* b_+ - b_+ b_+^*) = 1 - q^2, \quad (3.12)$$

$$a_+ a_+^* + a_- a_-^* + b_+ b_+^* + b_- b_-^* = 1, \quad (3.13)$$

$$a_+^* a_+ + a_-^* a_- + q^2 (b_+^* b_+ + b_-^* b_-) = 1, \quad (3.14)$$

$$a_-^* a_- - q^2 a_- a_-^* + q^2 b_-^* b_- - q^2 b_- b_-^* = 0, \quad (3.15)$$

$$a_+ a_+^* + b_-^* b_+ = 0, \quad a_-^* a_+ + q^2 b_-^* b_+ = 0, \quad (3.16)$$

$$a_- a_+^* + b_+^* b_- = 0, \quad a_+^* a_- + q^2 b_+^* b_- = 0, \quad (3.17)$$

$$b_+ b_+^* - b_+^* b_+ + b_- b_-^* - b_-^* b_- = 0, \quad (3.18)$$

$$q a_+ b_- - b_- a_+ + q a_- b_+ - b_+ a_- = 0. \quad (3.19)$$

And two others:

Note that we also use two other infinite dimensional $*$ -representations π_{\pm} of \mathcal{A} on $\ell^2(\mathbb{N})$ defined as follows on the orthonormal basis $\{\varepsilon_n : n \in \mathbb{N}\}$ of $\ell^2(\mathbb{N})$ by

$$\pi_{\pm}(a) \varepsilon_n := q_{n+1} \varepsilon_{n+1}, \quad \pi_{\pm}(b) \varepsilon_n := \pm q^n \varepsilon_n. \quad (3.20)$$

These representations are irreducible but not faithful since for instance $\pi_{\pm}(b - b^*) = 0$.

The Dirac operator:

It is chosen as in the classical case of a 3-sphere with the round metric:

$$\mathcal{D} |j\mu n\rangle := \begin{pmatrix} 2j+\frac{3}{2} & 0 \\ 0 & -2j-\frac{1}{2} \end{pmatrix} |j\mu n\rangle, \quad (3.21)$$

which means, with our convention, that $\mathcal{D} v_{ml}^j = \begin{pmatrix} 2j+\frac{3}{2} & 0 \\ 0 & -2j-\frac{1}{2} \end{pmatrix} v_{ml}^j$. Note that this operator is invertible (and thus $D = \mathcal{D}, P_0 = 0$). Moreover, it is asymptotically diagonal with linear spectrum and

the eigenvalues $2j + \frac{1}{2}$ for $j \in \frac{1}{2}\mathbb{N}$, have multiplicities $(2j+1)(2j+2)$,
the eigenvalues $-(2j + \frac{1}{2})$ for $j \in \frac{1}{2}\mathbb{N}^*$, have multiplicities $2j(2j+1)$.

So this Dirac operator coincides exactly with the classical one on the 3-sphere (see [4, 80]) with a gap around 0.

Let $\mathcal{D} = F|\mathcal{D}|$ be the polar decomposition of \mathcal{D} , thus

$$|\mathcal{D}| |j\mu n\rangle = \begin{pmatrix} d_j+ & 0 \\ 0 & d_j \end{pmatrix} |j\mu n\rangle, \quad d_j := 2j + \frac{1}{2}, \quad (3.22)$$

$$F |j\mu n\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} |j\mu n\rangle, \quad (3.23)$$

and it follows from (3.7) and (3.23) that

$$F \text{ commutes with } a_{\pm}, b_{\pm}. \quad (3.24)$$

The reality operator:

This antilinear operator J is defined on the basis of \mathcal{H} by

$$J |j, \mu, n, \uparrow\rangle := i^{2(2j+\mu+n)} |j, -\mu, -n, \uparrow\rangle, \quad J |j, \mu, n, \downarrow\rangle := i^{2(2j-\mu-n)} |j, -\mu, -n, \downarrow\rangle \quad (3.25)$$

thus it satisfies

$$J^{-1} = -J = J^* \text{ and } \mathcal{D}J = J\mathcal{D},$$

$$J v_{m,l}^{j\uparrow} = i^{2(m+l)-1} v_{2j-m, 2j+1-l}^{j\uparrow}, \quad J v_{m,l}^{j\downarrow} = i^{-2(m+l)+1} v_{2j-m, 2j-1-l}^{j\downarrow}.$$

We denote

\mathcal{B} the $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$ generated by the operators in $\delta^k(\pi(\mathcal{A}))$ for all $k \in \mathbb{N}$,

$\Psi_0^0(\mathcal{A})$ the algebra generated by $\delta^k(\pi(\mathcal{A}))$ and $\delta^k([\mathcal{D}, \pi(\mathcal{A})])$ for all $k \in \mathbb{N}$,

X the $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$ algebraically generated by the set $\{a_{\pm}, b_{\pm}\}$.

Note that $\Psi_0^0(\mathcal{A})$ is a subalgebra of $\Psi^0(\mathcal{A})$ (the space of pseudodifferential operators of order less or equal to zero).

The Hopf map r

For the explicit calculations of residues, we need a $*$ -homomorphism $r : X \rightarrow \pi_+(\mathcal{A}) \otimes \pi_-(\mathcal{A})$ defined by the tensor product in the sense of Hopf algebras of representations π_+ and π_- :

$$\begin{aligned} r(a_+) &:= \pi_+(a) \otimes \pi_-(a), & r(a_-) &:= -q \pi_+(b) \otimes \pi_-(b^*), \\ r(b_+) &:= -\pi_+(a) \otimes \pi_-(b), & r(b_-) &:= -\pi_+(b) \otimes \pi_-(a^*). \end{aligned} \quad (3.26)$$

In fact, \mathcal{A} is a Hopf * -algebra under the coproduct $\Delta(a) := a \otimes a - q b \otimes b^*$, $\Delta(b) := a \otimes b + b \otimes a^*$. These homomorphisms appeared in [153] with the translation $\alpha \leftrightarrow a^*$, $\gamma \leftrightarrow -b$. In particular, if $U := \begin{pmatrix} a & b \\ -qb^* & a^* \end{pmatrix}$ is the canonical generator of the $K_1(\mathcal{A})$ -group $(\Delta a, \Delta b) = (a, b) \dot{\otimes} U$ where the last $\dot{\otimes}$ means the matrix product of tensors of components.

The grading:

According to the shift $j \rightarrow j^\pm$ appearing in formulae (3.9), (3.10), we get a \mathbb{Z} -grading on X defined by the degree +1 on a_+, b_+, a_-^*, b_-^* and -1 on a_-, b_-, a_+^*, b_+^* .

Any operator $T \in X$ can be (uniquely) decomposed as $T = \sum_{j \in \mathbb{Z}} T_j$ where T_j is homogeneous of degree j .

For $T \in X$, T° will denote the 0-degree part of T for this grading and by a slight abuse of notations, we write $r(T)^\circ$ instead of $r(T^\circ)$.

The symbol map:

We also use the * -homomorphism $\sigma: \pi_\pm(\mathcal{A}) \rightarrow C^\infty(S^1)$ defined for $z \in S^1$ on the generators by

$$\sigma(\pi_\pm(a))(z) := z, \quad \sigma(\pi_\pm(a^*))(z) := \bar{z}, \quad \sigma(\pi_\pm(b))(z) = \sigma(\pi_\pm(b^*))(z) := 0.$$

The application $(\sigma \otimes \sigma) \circ r$ is defined on X (and so on \mathcal{B}) with values in $C^\infty(S^1) \otimes C^\infty(S^1)$.

We define

$$dT := [\mathcal{D}, T] \text{ and } \delta(T) := [|\mathcal{D}|, T].$$

Lemma 3.3.2. a_\pm, b_\pm are bounded operators on \mathcal{H} such that for all $p \in \mathbb{N}$,

- (i) $\delta(a_\pm) = \pm a_\pm$, $\delta(b_\pm) = \pm b_\pm$,
- (ii) $\delta^p(\underline{\pi}(a)) = a_+ + (-1)^p a_-$, $\delta^p(\underline{\pi}(b)) = b_+ + (-1)^p b_-$,
- (iii) $\delta(a_\pm^p) = \pm p a_\pm^{p-1}$, $\delta(b_\pm^p) = \pm p b_\pm^{p-1}$.

Proof. (i) By definition, $a_\pm |j\mu n\rangle = \begin{pmatrix} \alpha_\pm & 0 \\ 0 & \beta_\pm \end{pmatrix} |j^\pm \mu^+ n^+\rangle$ where the numbers α_\pm and β_\pm depend on j, μ, n and q , so we get by (3.22)

$$\begin{aligned} \delta(a_\pm) |j\mu n\rangle &= \begin{pmatrix} (d_{j^\pm})^{\alpha_\pm} & 0 \\ 0 & d_{j^\pm} \beta_\pm \end{pmatrix} |j^\pm \mu^+ n^+\rangle - \begin{pmatrix} (d_{j^\pm})^{\alpha_\pm} & 0 \\ 0 & d_{j^\pm} \beta_\pm \end{pmatrix} |j^\pm \mu^+ n^+\rangle \\ &= \begin{pmatrix} \pm \alpha_\pm & 0 \\ 0 & \pm \beta_\pm \end{pmatrix} |j^\pm \mu^+ n^+\rangle = \pm a_\pm |j\mu n\rangle \end{aligned}$$

and similar proofs for b_\pm .

(ii) and (iii) are straightforward consequences of (i) and definition of $\underline{\pi}$. □

Remark 3.3.3. By Lemma 3.3.2, we see that, modulo $OP^{-\infty}$, X is equal to \mathcal{B} and in particular contains $\pi(\mathcal{A})$.

Using (3.24), we get that $\mathcal{B} \subset \Psi_0^0(\mathcal{A}) \subset$ algebra generated by \mathcal{B} and $\mathcal{B}F$.

Note that, despite the last inclusion, F is not a priori in $\Psi_0^0(\mathcal{A})$.

3.3.2 The noncommutative integrals

Recall that for any pseudodifferential operator T , $\oint T := \text{Res}_{s=0} \zeta_D^T(s)$ where $\zeta_D^T(s) := \text{Tr}(T|\mathcal{D}|^{-s})$.

Theorem 3.3.4. *The dimension spectrum (without reality structure given by J) of the spectral triple $(\mathcal{A}(SU_q(2)), \mathcal{H}, \mathcal{D})$ is simple and equal to $\{1, 2, 3\}$.*

Moreover, the corresponding residues for $T \in \mathcal{B}$ are

$$\begin{aligned} \oint T|\mathcal{D}|^{-3} &= 2(\tau_1 \otimes \tau_1)(r(T)^\circ), \\ \oint T|\mathcal{D}|^{-2} &= 2(\tau_1 \otimes \tau_0 + \tau_0 \otimes \tau_1)(r(T)^\circ), \\ \oint T|\mathcal{D}|^{-1} &= (2\tau_0 \otimes \tau_0 - \tfrac{1}{2}\tau_1 \otimes \tau_1)(r(T)^\circ), \\ \oint FT|\mathcal{D}|^{-3} &= 0, \\ \oint FT|\mathcal{D}|^{-2} &= 0, \\ \oint FT|\mathcal{D}|^{-1} &= (\tau_0 \otimes \tau_1 - \tau_1 \otimes \tau_0)(r(T)^\circ), \end{aligned}$$

where the functionals τ_0, τ_1 are defined for $x \in \pi_\pm(\mathcal{A})$ by

$$\tau_0(x) := \lim_{N \rightarrow \infty} (\text{Tr}_N x - (N+1)\tau_1(x)), \quad \tau_1(x) := \frac{1}{2\pi} \int_0^{2\pi} \sigma(x)(e^{i\theta}) d\theta,$$

with $\text{Tr}_N x = \sum_{n=0}^N \langle \varepsilon_n, x \varepsilon_n \rangle$.

Proof. Consequence of [138, Theorem 4.1 and (4.3)]. \square

Remark 3.3.5. Since F is not in \mathcal{B} , the equations of Theorem 3.3.4 are not valid for all $T \in \Psi_0^0(\mathcal{A})$.

But when $T \in \Psi_0^0(\mathcal{A})$, $\oint T|\mathcal{D}|^{-k} = 0$ for $k \notin \{1, 2, 3\}$ since the dimension spectrum is $\{1, 2, 3\}$ [138].

Compared to [138] where we had

$$\tau_0^\uparrow(x) := \lim_{N \rightarrow \infty} \text{Tr}_N x - (N + \tfrac{3}{2})\tau_1(x), \quad \tau_0^\downarrow(x) := \lim_{N \rightarrow \infty} \text{Tr}_N x - (N + \tfrac{1}{2})\tau_1(x),$$

we replaced them with τ_0 :

$$\tau_0^\uparrow = \tau_0 - \tfrac{1}{2}\tau_1, \quad \tau_0^\downarrow = \tau_0 + \tfrac{1}{2}\tau_1.$$

Note that τ_1 is a trace on $\pi_\pm(\mathcal{A})$ such that $\tau_1(1) = 1$ and $\tau_1(\pi_+(aa^*)) = \frac{1}{2\pi} \int_0^{2\pi} 1 d\theta = 1$, while τ_0 is not since $\tau_0(1) = 0$ and

$$\tau_0(\pi_\pm(aa^*)) = \lim_{N \rightarrow \infty} \sum_{n=0}^N (1 - q^{2n}) - (N+1) = -\frac{1}{1-q^2}, \quad (3.27)$$

so, because of the shift, the substitution $a \leftrightarrow a^*$ gives

$$\tau_0(\pi_\pm(a^*a)) = q^2 \tau_0(\pi_\pm(aa^*)). \quad (3.28)$$

3.3.3 The tadpole

Lemma 3.3.6. *For $SU_q(2)$, the condition of the vanishing tadpole (see [37]) is not satisfied.*

Proof. For example, an explicit calculation gives $\oint \pi(b)[\mathcal{D}, \pi(b^*)]\mathcal{D}^{-1} = \frac{2}{1-q^2}$:

Let $x, y \in \underline{\pi}(\mathcal{A})$. Since $[F, x] = 0$, we have

$$\oint x[\mathcal{D}, y]\mathcal{D}^{-1} = \oint x\delta(y)|\mathcal{D}|^{-1} = \tau'(r(x\delta(y)))^0$$

where $\tau' := 2\tau_0 \otimes \tau_0 - \frac{1}{2}\tau_1 \otimes \tau_1$.

By Lemma 3.3.2, $\underline{\pi}(b)\delta(\underline{\pi}(b^*)) = (b_+ + b_-)((b_-)^* - (b_+)^*) = -b_+b_+^* + b_-b_-^* + b_+b_-^* - b_-b_+^*$. Since only the first two terms have degree 0, we get, using the formulae from Theorem 3.3.4

$$\begin{aligned} \tau'(r(-b_+b_+^*)) &= -\tau'(\pi_+(aa^*) \otimes \pi_-(bb^*)) \\ &= -2\tau_0(\pi_+(aa^*))\tau_0(\pi_-(bb^*)) + \frac{1}{2}\tau_1(\pi_+(aa^*))\tau_1(\pi_-(bb^*)) \end{aligned}$$

and $\tau_1(\pi_-(bb^*)) = 0$. Similarly, using (3.28)

$$\tau'(r(b_-b_-^*)) = 2\tau_0(\pi_+(bb^*))\tau_0(\pi_-(a^*a)) = 2q^2\tau_0(\pi_-(aa^*))\tau_0(\pi_+(bb^*)).$$

Since $\tau_0(\pi_{\pm}(bb^*)) = \text{Tr}(\pi_{\pm}(bb^*)) = \sum_{n=0}^{\infty} q^{2n} = \frac{1}{1-q^2}$ and (3.27),

$$\oint \pi(b)[\mathcal{D}, \pi(b^*)]\mathcal{D}^{-1} = 2\frac{1}{1-q^2}\frac{1}{1-q^2} + 2q^2\frac{-1}{1-q^2}\frac{1}{1-q^2} = \frac{2}{1-q^2}. \quad \square$$

In particular the pairing of the tadpole cyclic cocycle ϕ_1 with the generator of K_1 -group is nontrivial:

Remark 3.3.7. *Other examples: with the shorthand x instead of $\underline{\pi}(x)$,*

$$\begin{aligned} (\tau_1 \otimes \tau_1) r(a\delta(a^*)^\circ) &= -1, & (\tau_1 \otimes \tau_1) r(a^*\delta(a)^\circ) &= 1, \\ (\tau_0 \otimes \tau_0) r(a\delta(a^*)^\circ) &= \frac{1}{q^2-1}, & (\tau_0 \otimes \tau_0) r(a^*\delta(a)^\circ) &= \frac{q^2}{q^2-1}, \\ \oint a\delta(a^*)|\mathcal{D}|^{-1} &= \frac{q^2+3}{2(q^2-1)}, & \oint a^*\delta(a)|\mathcal{D}|^{-1} &= \frac{3q^2+1}{2(q^2-1)}, \\ \oint b\delta(b)|\mathcal{D}|^{-1} &= 0, & \oint b^*\delta(b^*)|\mathcal{D}|^{-1} &= 0, \\ \oint b\delta(b^*)|\mathcal{D}|^{-1} &= \frac{-2}{q^2-1}, & \oint b^*\delta(b)|\mathcal{D}|^{-1} &= \frac{-2}{q^2-1}. \end{aligned}$$

In particular, $N\phi_1$ does not vanish on 1-forms since $\int_{N\phi_1} ada^* = N\phi_1(a, a^*) = -1$.

Let U be the canonical generator of the $K_1(\mathcal{A})$ -group, $U = \begin{pmatrix} a & b \\ -qb^* & a^* \end{pmatrix}$ acting on $\mathcal{H} \otimes \mathbb{C}^2$. Then for $A_U := \sum_{k,l=1}^2 \pi(U_{kl}) d\pi(U^*_{kl})$, using above remark, $\int_{\phi_1} A_U = -2$ as obtained in [138, page 391]: in fact, with $P := \frac{1}{2}(1 + F)$,

$$\psi_1(U, U^*) := 2 \sum_{k,l} \oint U_{kl} \delta(U^*_{kl}) P |\mathcal{D}|^{-1} - \oint U_{kl} \delta^2(U^*_{kl}) P |\mathcal{D}|^{-2} + \frac{2}{3} \oint U_{kl} \delta^3(U^*_{kl}) P |\mathcal{D}|^{-3}$$

satisfies $\psi_1(U, U^*) = 2 \sum_{k,l} \oint U_{kl} \delta(U^*_{kl}) P |\mathcal{D}|^{-1} = \int_{\phi_1} A_U$.

3.4 Reality operator and spectral action on $SU_q(2)$

3.4.1 Spectral action in dimension 3 with $[F, \mathcal{A}] \in OP^{-\infty}$

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a real spectral triple of dimension 3. Assume that $[F, \mathcal{A}] \in OP^{-\infty}$, where $F := \mathcal{D}|\mathcal{D}|^{-1}$ (we suppose \mathcal{D} invertible). Let \mathbb{A} be a selfadjoint one form, so \mathbb{A} is of the form $\mathbb{A} = \sum_i a_i db_i$ where $a_i, b_i \in \mathcal{A}$.

Thus, $\mathbb{A} \simeq AF \mod OP^{-\infty}$ where $A := \sum_i a_i \delta(b_i)$ is the δ -one-form associated to \mathbb{A} . Note that A and F commute modulo $OP^{-\infty}$.

We define

$$\begin{aligned} D_{\mathbb{A}} &:= \mathcal{D}_{\mathbb{A}} + P_{\mathbb{A}}, \quad P_{\mathbb{A}} \text{ the projection on } \text{Ker } \mathcal{D}_{\mathbb{A}}, \\ \mathcal{D}_{\mathbb{A}} &:= \mathcal{D} + \tilde{\mathbb{A}}, \quad \tilde{\mathbb{A}} := \mathbb{A} + J\mathbb{A}J^{-1}. \end{aligned}$$

Theorem 3.4.1. *The coefficients of the full spectral action (with reality operator) on any real spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of dimension 3 such that $[F, \mathcal{A}] \in OP^{-\infty}$ are*

$$\begin{aligned} (i) \quad & \int |D_{\mathbb{A}}|^{-3} = \int |\mathcal{D}|^{-3}. \\ (ii) \quad & \int |D_{\mathbb{A}}|^{-2} = \int |\mathcal{D}|^{-2} - 4 \int A |\mathcal{D}|^{-3}. \\ (iii) \quad & \int |D_{\mathbb{A}}|^{-1} = \int |\mathcal{D}|^{-1} - 2 \int A |\mathcal{D}|^{-2} + 2 \int A^2 |\mathcal{D}|^{-3} + 2 \int AJAJ^{-1} |\mathcal{D}|^{-3}. \\ (iv) \quad & \zeta_{D_{\mathbb{A}}}(0) = \zeta_{\mathcal{D}}(0) - 2 \int A |\mathcal{D}|^{-1} + \int A(A + JAJ^{-1}) |\mathcal{D}|^{-2} + \int \delta(A)(A + JAJ^{-1}) |\mathcal{D}|^{-3} \\ & \quad - \frac{2}{3} \int A^3 |\mathcal{D}|^{-3} - 2 \int A^2 JAJ^{-1} |\mathcal{D}|^{-3}. \end{aligned}$$

Proof. (i) We apply Proposition 1.3.9.

(ii) By Lemma 1.3.10, we have $\int |D_{\mathbb{A}}|^{-2} = \int |\mathcal{D}|^{-2} - \int (\tilde{\mathbb{A}}\mathcal{D} + \mathcal{D}\tilde{\mathbb{A}} + \tilde{\mathbb{A}}^2) |\mathcal{D}|^{-4}$. By the trace property of the noncommutative integral and the fact that $\tilde{\mathbb{A}}^2 |\mathcal{D}|^{-4}$ is trace-class, we get $\int |D_{\mathbb{A}}|^{-2} = \int |\mathcal{D}|^{-2} - 2 \int \tilde{\mathbb{A}}\mathcal{D} |\mathcal{D}|^{-4} = \int |\mathcal{D}|^{-2} - 4 \int \mathbb{A}\mathcal{D} |\mathcal{D}|^{-4}$. Since $\mathbb{A}\mathcal{D} \sim A|\mathcal{D}| \mod OP^{-\infty}$, we get the result.

(iii) By Lemma 1.3.10 (ii), we have

$$\int |D_{\mathbb{A}}|^{-1} = \int |\mathcal{D}|^{-1} - \frac{1}{2} \int (\tilde{\mathbb{A}}\mathcal{D} + \mathcal{D}\tilde{\mathbb{A}} + \tilde{\mathbb{A}}^2) |\mathcal{D}|^{-3} + \frac{3}{8} \int (\tilde{\mathbb{A}}\mathcal{D} + \mathcal{D}\tilde{\mathbb{A}} + \tilde{\mathbb{A}}^2)^2 |\mathcal{D}|^{-5}.$$

Following arguments of (ii), we get

$$\begin{aligned} \int (\tilde{\mathbb{A}}\mathcal{D} + \mathcal{D}\tilde{\mathbb{A}} + \tilde{\mathbb{A}}^2) |\mathcal{D}|^{-3} &= 4 \int A |\mathcal{D}|^{-2} + 2 \int A^2 |\mathcal{D}|^{-3} + 2 \int AJAJ^{-1} |\mathcal{D}|^{-3}, \\ \int (\tilde{\mathbb{A}}\mathcal{D} + \mathcal{D}\tilde{\mathbb{A}} + \tilde{\mathbb{A}}^2)^2 |\mathcal{D}|^{-5} &= 8 \int A^2 |\mathcal{D}|^{-3} + 8 \int AJAJ^{-1} |\mathcal{D}|^{-3}, \end{aligned}$$

and the result follows.

(iv) By (1.11), $\zeta_{D_{\mathbb{A}}}(0) = \sum_{j=1}^3 \frac{(-1)^j}{j} \int \tilde{\mathbb{A}}^j |\mathcal{D}|^{-j-1}$.

Moreover, the following holds: $\int \tilde{\mathbb{A}} |\mathcal{D}|^{-1} = 2 \int A |\mathcal{D}|^{-1}$ and $\int \tilde{\mathbb{A}}^2 |\mathcal{D}|^{-3} = 2 \int A^2 |\mathcal{D}|^{-3} + 2 \int AJAJ^{-1} |\mathcal{D}|^{-3}$. Since $\delta(A) \in OP^0$, we can check that $\int \delta(A) |\mathcal{D}|^{-1} = \int A^2 |\mathcal{D}|^{-2} +$

$\oint \delta(A)A|\mathcal{D}|^{-3}$ and, with the same argument, that $\oint A|\mathcal{D}|^{-1}JAJ^{-1}|\mathcal{D}|^{-1} = \oint AJAJ^{-1}|\mathcal{D}|^{-2} + \oint \delta(A)JAJ^{-1}|\mathcal{D}|^{-3}$. Thus, we get

$$\oint (\tilde{\mathbb{A}}\mathcal{D}^{-1})^2 = 2 \oint A(A + JAJ^{-1})|\mathcal{D}|^{-2} + 2 \oint \delta(A)(A + JAJ^{-1})|\mathcal{D}|^{-3}. \quad (3.29)$$

The third term to be computed is

$$\oint (\tilde{\mathbb{A}}\mathcal{D}^{-1})^3 = 2 \oint (A|\mathcal{D}|^{-1})^3 + 4 \oint (A|\mathcal{D}|^{-1})^2 JAJ^{-1}|\mathcal{D}|^{-1} + 2 \oint A|\mathcal{D}|^{-1} JAJ^{-1}|\mathcal{D}|^{-1} A|\mathcal{D}|^{-1}.$$

Any operator in OP^{-4} being trace-class here, we get

$$\oint (\tilde{\mathbb{A}}\mathcal{D}^{-1})^3 = 2 \oint A^3|\mathcal{D}|^{-3} + 4 \oint A^2 JAJ^{-1}|\mathcal{D}|^{-3} + 2 \oint AJAJ^{-1}A|\mathcal{D}|^{-3}. \quad (3.30)$$

Since $\oint AJAJ^{-1}A|\mathcal{D}|^{-3} = \oint A^2 JAJ^{-1}|\mathcal{D}|^{-3}$ by trace property and the fact that $\delta(A) \in OP^0$, the result follows then from (3.29) and (3.30). \square

Corollary 3.4.2. *For the spectral action of \mathbb{A} without the reality operator (i.e. $\mathcal{D}_{\mathbb{A}} = \mathcal{D} + \mathbb{A}$), we get*

$$\begin{aligned} \oint |D_{\mathbb{A}}|^{-2} &= \oint |\mathcal{D}|^{-2} - 2 \oint A|\mathcal{D}|^{-3}, \\ \oint |D_{\mathbb{A}}|^{-1} &= \oint |\mathcal{D}|^{-1} - \oint A|\mathcal{D}|^{-2} + \oint A^2|\mathcal{D}|^{-3}, \\ \zeta_{D_{\mathbb{A}}}(0) &= \zeta_{\mathcal{D}}(0) - \oint A|\mathcal{D}|^{-1} + \frac{1}{2} \oint A^2|\mathcal{D}|^{-2} + \frac{1}{2} \oint \delta(A)A|\mathcal{D}|^{-3} - \frac{1}{3} \oint A^3|\mathcal{D}|^{-3}. \end{aligned}$$

3.4.2 Spectral action on $SU_q(2)$: main result

On $SU_q(2)$, since F commutes with a_{\pm} and b_{\pm} , Theorem 3.4.1 can be used for the spectral action computation.

Here is the main result of this section

Theorem 3.4.3. *In the full spectral action (1.6) (with the reality operator) of $SU_q(2)$ for a one-form \mathbb{A} and A its associated δ -one-form, the coefficients are:*

$$\begin{aligned} \oint |D_{\mathbb{A}}|^{-3} &= 2, \\ \oint |D_{\mathbb{A}}|^{-2} &= -4 \oint A|\mathcal{D}|^{-3}, \\ \oint |D_{\mathbb{A}}|^{-1} &= -\frac{1}{2} + 2 \left(\oint A^2|\mathcal{D}|^{-3} - \oint A|\mathcal{D}|^{-2} \right) + \left| \oint A|\mathcal{D}|^{-3} \right|^2, \\ \zeta_{D_{\mathbb{A}}}(0) &= -2 \oint A|\mathcal{D}|^{-1} + \oint A^2|\mathcal{D}|^{-2} - \frac{2}{3} \oint A^3|\mathcal{D}|^{-3} \\ &\quad + \overline{\oint A|\mathcal{D}|^{-3}} \left(\frac{1}{2} \oint A|\mathcal{D}|^{-2} - \oint A^2|\mathcal{D}|^{-3} \right) + \frac{1}{2} \oint A|\mathcal{D}|^{-3} \overline{\oint A|\mathcal{D}|^{-2}}. \end{aligned}$$

In order to prove this theorem (in section 4.7), we will use a decomposition of one-forms in the Poincaré-Birkhoff-Witt basis of \mathcal{A} with an extension of previous representations to operators like $TJT'J^{-1}$ where T and T' are in X .

3.4.3 Balanced components and Poincaré–Birkhoff–Witt basis of \mathcal{A}

Our objective is to compute all integrals in term of A and the computation will lead to functions of A which capture certain symmetries on A .

Let $\mathbb{A} = \sum_i \pi(x^i) d\pi(y^i)$ on $SU_q(2)$ be a one-form and A the associated δ -one-form. The x^i and y^i are in \mathcal{A} and as such they can be uniquely written as finite sums $x^i = \sum_\alpha x_\alpha^i m^\alpha$ and $y^i = \sum_\beta y_\beta^i m^\beta$ where $m^\alpha := a^{\alpha_1} b^{\alpha_2} b^{*\alpha_3}$ is the canonical monomial of \mathcal{A} with $\alpha, \beta \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N}$ based on a fixed Poincaré–Birkhoff–Witt type basis of \mathcal{A} .

Remark 3.4.4. Any one-form $\mathbb{A} = \sum_i \pi(x^i) d\pi(y^i)$ on $SU_q(2)$ is characterized by a complex valued matrix $A_\alpha^\beta = \sum_i x_\alpha^i y_\beta^i$ where $\alpha, \beta \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N}$. This matrix is such that

$$A = A_\alpha^\beta M_\beta^\alpha$$

where $M_\beta^\alpha := \pi(m^\alpha) \delta(\pi(m^\beta))$.

In the following, we note

$$\bar{A} := \bar{A}_\alpha^\beta M_\beta^\alpha$$

so for any $p \in \mathbb{N}$, $\int \bar{A} |\mathcal{D}|^{-p} = \overline{\int A |\mathcal{D}|^{-p}}$.

This presentation of one-forms is not unique modulo $OP^{-\infty}$ since, as we will see in section 5, $F = \sum_i x_i dy_i$ where $x_i, y_i \in \mathcal{A}$, thus for any generator z , $[F, z] = \sum_i x_i d(y_i z) - x_i y_i dz - z x_i dy_i = 0 \mod OP^{-\infty}$. We do not know however if this presentation is unique when the $OP^{-\infty}$ part is taken into account.

The δ -one-forms M_β^α are said to be *canonical*. Any product of n canonical δ -one forms, where $n \in \mathbb{N}^*$, is called a *canonical δ^n -one-form*. Thus, if A is a δ -one-form, $A^n = (A^n)_{\bar{\alpha}}^{\bar{\beta}} M_{\bar{\beta}}^{\bar{\alpha}}$ where $\bar{\alpha} = (\alpha, \alpha', \dots, \alpha^{(n-1)})$, $\bar{\beta} = (\beta, \beta', \dots, \beta^{(n-1)})$ are in $\mathbb{Z}^n \times \mathbb{N}^n \times \mathbb{N}^n$, $(A^n)_{\bar{\alpha}}^{\bar{\beta}} := A_\alpha^\beta \dots A_{\alpha^{(n-1)}}^{\beta^{(n-1)}}$ and $M_{\bar{\beta}}^{\bar{\alpha}}$ is the canonical δ^n -one form equal to $M_\beta^\alpha \dots M_{\beta^{(n-1)}}^{\alpha^{(n-1)}}$.

Definition 3.4.5. A canonical δ^n -one-form is *a-balanced* if it is of the form

$$a^{\alpha_1} \delta(a^{\beta_1}) \dots a^{\alpha_1^{(n-1)}} \delta(a^{\beta_1^{(n-1)}})$$

where $\sum_{i=0}^{n-1} \alpha_1^{(i)} + \beta_1^{(i)} = 0$.

For any δ -one-form A , the *a-balanced* components of A^n are denoted $B_a(A^n)_{\bar{\alpha}}^{\bar{\beta}}$.

Note that

$$B_a(A)_{\bar{\alpha}}^{\bar{\beta}} = A_{-\beta_1 00}^{\beta_1 00} \delta_{\alpha_1 + \beta_1, 0} \delta_{\alpha_2 + \alpha_3 + \beta_2 + \beta_3, 0}.$$

Definition 3.4.6. A canonical δ^n -one-form is *balanced* if it is of the form

$$m^\alpha \delta(m^\beta) \dots m^{\alpha^{(n-1)}} \delta(m^{\beta^{(n-1)}})$$

where $\sum_{i=0}^{n-1} \alpha_1^{(i)} + \beta_1^{(i)} = 0$ and $\sum_{i=0}^{n-1} \alpha_2^{(i)} + \beta_2^{(i)} = \sum_{i=0}^{n-1} \alpha_3^{(i)} + \beta_3^{(i)}$.

For any δ -one-form A , the *balanced* components of A^n are denoted $B(A^n)_{\bar{\alpha}}^{\bar{\beta}}$.

Note that

$$B(A)_{\bar{\alpha}}^{\bar{\beta}} = A_{-\beta_1\alpha_2\alpha_3}^{\beta_1\beta_2\beta_3} \delta_{\alpha_1+\beta_1,0} \delta_{\alpha_2+\beta_2,\alpha_3+\beta_3}.$$

As we will show, a contribution to the k^{th} -coefficient in the spectral action, is only brought by one-forms \mathbb{A} such that A^k is balanced (and even a -balanced in the case $k = 1$).

Note also that if A is balanced, then A^k for $k \geq 1$ is also balanced, whereas the converse is false.

3.4.4 The reality operator J on $SU_q(2)$

Let for any $n, p \in \mathbb{N}$,

$$\begin{aligned} q_n &:= \sqrt{1 - q^{2n}}, & q_{-n} &:= 0 \text{ if } n > 0, \\ q_{n,p}^{\uparrow} &:= q_{n+1} \cdots q_{n+p}, & q_{n,p}^{\downarrow} &:= q_n \cdots q_{n-(p-1)}, \end{aligned}$$

with the convention $q_{n,0}^{\uparrow} = q_{n,0}^{\downarrow} := 1$. Thus, we have the relations

$$\begin{aligned} \pi_{\pm}(a^p) \varepsilon_n &= q_{n,p}^{\uparrow} \varepsilon_{n+p}, & \pi_{\pm}(a^{*p}) \varepsilon_n &= q_{n,p}^{\downarrow} \varepsilon_{n-p}, \\ \pi_{\pm}(b^p) \varepsilon_n &= (\pm q^n)^p \varepsilon_n, & \pi_{\pm}(b^{*p}) \varepsilon_n &= (\pm q^n)^p \varepsilon_n, \end{aligned}$$

where $\varepsilon_k := 0$ if $k < 0$.

The sign of $x \in \mathbb{R}$ is denoted η_x . By convention, $a_j := a$, $a_{\pm,j} := a_{\pm}$ if $j \geq 0$ and $a_j := a^*$, $a_{\pm,j} := a_{\pm}^*$ if $j < 0$. Note that, with convention

$$q_{n,p}^{\uparrow\alpha_1} := q_{n,p}^{\uparrow} \text{ if } \alpha_1 > 0, \quad q_{n,p}^{\downarrow\alpha_1} := q_{n,p}^{\downarrow} \text{ if } \alpha_1 < 0, \text{ and } q_{n,p}^{\uparrow 0} := 1,$$

we have for any $\alpha_1 \in \mathbb{Z}$ and $p \leq \alpha_1$, $\pi_{\pm}(a_{\alpha_1}^p) \varepsilon_n = q_{n,p}^{\uparrow\alpha_1} \varepsilon_{n+\eta_{\alpha_1}p}$.

Recall that the reality operator J is defined by

$$J v_{m,l}^{j\uparrow} = i^{2(m+l)-1} v_{2j-m,2j+1-l}^{j\uparrow}, \quad J v_{m,l}^{j\downarrow} = i^{-2(m+l)+1} v_{2j-m,2j-1-l}^{j\downarrow},$$

thus the real conjugate operators

$$\widehat{a}_{\pm} := J a_{\pm} J^{-1}, \quad \widehat{b}_{\pm} := J b_{\pm} J^{-1}$$

satisfy

$$\begin{aligned} \widehat{a}_+ v_{m,l}^j &:= -q_{2j+1-m} \begin{pmatrix} q_{2j+2-l} & 0 \\ 0 & q_{2j-l} \end{pmatrix} v_{m,l}^{j+}, & \widehat{a}_- v_{m,l}^j &:= -q^{2j-m} \begin{pmatrix} q^{2j+2-l} & 0 \\ 0 & q^{2j-l} \end{pmatrix} v_{m-1,l-1}^{j-}, \\ \widehat{b}_+ v_{m,l}^j &:= q_{2j+1-m} \begin{pmatrix} q^{2j+1-l} & 0 \\ 0 & q^{2j-1-l} \end{pmatrix} v_{m,l+1}^{j+}, & \widehat{b}_- v_{m,l}^j &:= -q^{2j-m} \begin{pmatrix} q_{2j+1-l} & 0 \\ 0 & q_{2j-1-l} \end{pmatrix} v_{m-1,l}^{j-}. \end{aligned}$$

So the real conjugate operator behaves differently on the up and down part of the Hilbert space. The difference comes from the fact that the index l is not treated uniformly by J on up and down parts.

We denote \widehat{X} the algebra generated by $\{\widehat{a}_{\pm}, \widehat{b}_{\pm}\}$, \widetilde{X} the algebra generated by $\{a_{\pm}, b_{\pm}, \widehat{a}_{\pm}, \widehat{b}_{\pm}\}$ and $\mathcal{H}' := \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{Z})$ and we construct two $*$ -representations $\widehat{\pi}_{\pm}$ of \mathcal{A} :

The representation $\widehat{\pi}_+$ gives bounded operators on \mathcal{H}' while $\widehat{\pi}_-$ represents \mathcal{A} into $\mathcal{B}(\mathcal{H}' \otimes \mathbb{C}^2)$.

The representation $\hat{\pi}_+$ is defined on the generators by:

$$\hat{\pi}_+(a) \varepsilon_m \otimes \varepsilon_{2j} := q_{2j+1-m} \varepsilon_m \otimes \varepsilon_{2j+1}, \quad \hat{\pi}_+(b) \varepsilon_m \otimes \varepsilon_{2j} := -q^{2j-m} \varepsilon_{m+1} \otimes \varepsilon_{2j+1}$$

while $\hat{\pi}_-$ is defined by:

$$\begin{aligned} \hat{\pi}_-(a) \varepsilon_l \otimes \varepsilon_{2j} \otimes \varepsilon_{\uparrow\downarrow} &:= -q_{2j+1\pm 1-l} \varepsilon_l \otimes \varepsilon_{2j+1} \otimes \varepsilon_{\uparrow\downarrow}, \\ \hat{\pi}_-(b) \varepsilon_l \otimes \varepsilon_{2j} \otimes \varepsilon_{\uparrow\downarrow} &:= -q^{2j\pm 1-l} \varepsilon_{l+1} \otimes \varepsilon_{2j+1} \otimes \varepsilon_{\uparrow\downarrow}, \end{aligned}$$

where $\varepsilon_{\uparrow\downarrow}$ is the canonical basis of \mathbb{C}^2 and the $+$ in \pm corresponds to \uparrow in $\uparrow\downarrow$.

The link between $\hat{\pi}_\pm$ and π_\pm which explains the notations about these intermediate objects and the fact that $\hat{\pi}_\pm$ are representations on different Hilbert spaces, is in the parallel between equations (3.26), (3.31) and (3.32).

Let us give immediately a few properties (x_β equals x if the sign β is positive and equals x^* otherwise)

$$\begin{aligned} \hat{\pi}_+(a_\beta)^p \varepsilon_m \otimes \varepsilon_{2j} &= q_{2j-m,p}^{\uparrow\beta} \varepsilon_m \otimes \varepsilon_{2j+\eta_\beta p}, \\ \hat{\pi}_-(a_\beta)^p \varepsilon_l \otimes \varepsilon_{2j} \otimes \varepsilon_{\uparrow\downarrow} &= (-1)^p q_{2j\pm 1-l,p}^{\uparrow\beta} \varepsilon_l \otimes \varepsilon_{2j+\eta_\beta p} \otimes \varepsilon_{\uparrow\downarrow}, \\ \hat{\pi}_+(b_\beta)^p \varepsilon_m \otimes \varepsilon_{2j} &= (-1)^p q^{(2j-m)p} \varepsilon_{m+\eta_\beta p} \otimes \varepsilon_{2j+\eta_\beta p}, \\ \hat{\pi}_-(b_\beta)^p \varepsilon_l \otimes \varepsilon_{2j} \otimes \varepsilon_{\uparrow\downarrow} &= (-1)^p q^{(2j\pm 1-l)p} \varepsilon_{l+\eta_\beta p} \otimes \varepsilon_{2j+\eta_\beta p} \otimes \varepsilon_{\uparrow\downarrow}. \end{aligned}$$

Note that the $\hat{\pi}_\pm$ representations still contain the shift information, contrary to representations π_\pm . Moreover, $\hat{\pi}_\pm(b) \neq \hat{\pi}_\pm(b^*)$ while $\pi_\pm(b) = \pi_\pm(b^*)$.

The operators \hat{a}_\pm, \hat{b}_\pm are coded on $\mathcal{H}' \otimes \mathcal{H}' \otimes \mathbb{C}^2$ as the correspondence

$$\begin{aligned} \hat{a}_+ &\longleftrightarrow \hat{\pi}_+(a) \otimes \hat{\pi}_-(a), & \hat{a}_- &\longleftrightarrow -q \hat{\pi}_+(b^*) \otimes \hat{\pi}_-(b^*), \\ \hat{b}_+ &\longleftrightarrow -\hat{\pi}_+(a) \otimes \hat{\pi}_-(b), & \hat{b}_- &\longleftrightarrow -\hat{\pi}_+(b^*) \otimes \hat{\pi}_-(a^*). \end{aligned} \quad (3.31)$$

We now set the following extension to $\mathcal{B}(\mathcal{H}')$ of π_+ and to $\mathcal{B}(\mathcal{H}' \otimes \mathbb{C}^2)$ of π_- by

$$\begin{aligned} \pi'_+(a) &:= \pi_+(a) \otimes V, & \pi'_+(b) &:= \pi_+(b) \otimes V \quad (V \text{ is the shift of } \ell^2(\mathbb{Z})), \\ \pi'_-(a) &:= \pi_-(a) \otimes V \otimes 1_2, & \pi'_-(b) &:= \pi_-(b) \otimes V \otimes 1_2. \end{aligned}$$

So, we can define a canonical algebra morphism $\tilde{\rho}$ from \tilde{X} into the bounded operators on $\mathcal{H}' \otimes \mathcal{H}' \otimes \mathbb{C}^2$. This morphism is defined on the generators part $\{\hat{a}_\pm, \hat{b}_\pm\}$ of \tilde{X} by preceding correspondence and on the generators part $\{a_\pm, b_\pm\}$ by —see (3.26):

$$\begin{aligned} a_+ &\longleftrightarrow \pi'_+(a) \otimes \pi'_-(a), & a_- &\longleftrightarrow -q \pi'_+(b^*) \otimes \pi'_-(b^*), \\ b_+ &\longleftrightarrow -\pi'_+(a) \otimes \pi'_-(b), & b_- &\longleftrightarrow -\pi'_+(b^*) \otimes \pi'_-(a^*). \end{aligned} \quad (3.32)$$

We denote S the canonical surjection from $\mathcal{H}' \otimes \mathcal{H}' \otimes \mathbb{C}^2$ onto \mathcal{H} . This surjection is associated to the parameters restrictions on m, j, l, j' . In particular, the index j' associated to the second $\ell^2(\mathbb{N})$ in $\mathcal{H}' \otimes \mathcal{H}' \otimes \mathbb{C}^2$ is set to be equal to j . Any vector in $\mathcal{H}' \otimes \mathcal{H}' \otimes \mathbb{C}^2$ not satisfying these restrictions is sent to 0 in \mathcal{H} .

Denote by I the canonical injection of \mathcal{H} into $\mathcal{H}' \otimes \mathcal{H}' \otimes \mathbb{C}^2$ (the index j is doubled). Thus, $S\tilde{\rho}(\cdot)I$ is the identity on \tilde{X} .

In the computation of residues of ζ_D^T functions, we can therefore replace the operator T by $S\tilde{\rho}(T)I$.

We now extend τ_0 on $\pi'_\pm(\mathcal{A})\widehat{\pi}_\pm(\mathcal{A})$: For $x, y \in \mathcal{A}$, we set

$$\begin{aligned}\mathrm{Tr}_N(\pi'_+(x)\widehat{\pi}_+(y)) &:= \sum_{m=0}^N \langle \varepsilon_m \otimes \varepsilon_N, \pi'_+(x)\widehat{\pi}_+(y) \varepsilon_m \otimes \varepsilon_N \rangle, \\ \mathrm{Tr}_N^\dagger(\pi'_-(x)\widehat{\pi}_-(y)) &:= \sum_{l=0}^N \langle \varepsilon_l \otimes \varepsilon_{N-1} \otimes \varepsilon_\uparrow, \pi'_-(x)\widehat{\pi}_-(y) \varepsilon_l \otimes \varepsilon_{N-1} \otimes \varepsilon_\uparrow \rangle, \\ \mathrm{Tr}_N^\dagger(\pi'_-(x)\widehat{\pi}_-(y)) &:= \sum_{l=0}^N \langle \varepsilon_l \otimes \varepsilon_{N+1} \otimes \varepsilon_\downarrow, \pi'_-(x)\widehat{\pi}_-(y) \varepsilon_l \otimes \varepsilon_{N+1} \otimes \varepsilon_\downarrow \rangle.\end{aligned}$$

Actually, a computation on monomials of \mathcal{A} shows that $\mathrm{Tr}_N^\dagger(\pi'_-(x)\widehat{\pi}_-(y)) = \mathrm{Tr}_N^\dagger(\pi'_-(x)\widehat{\pi}_-(y))$. For convenience, we shall note $\mathrm{Tr}_N(\pi'_-(x)\widehat{\pi}_-(y))$ this functional.

Lemma 3.4.7. *Let $x, y \in \mathcal{A}$. Then,*

(i) $\tau_0(\pi'_\pm(x)\widehat{\pi}_\pm(y)) := \lim_{N \rightarrow \infty} U_N$ exists where

$$U_N := \mathrm{Tr}_N(\pi'_\pm(x)\widehat{\pi}_\pm(y)) - (N+1)\tau_1(\pi_\pm(x))\tau_1(\pi_\pm(y)).$$

(ii) $U_N = \tau_0(\pi'_\pm(x)\widehat{\pi}_\pm(y)) + \mathcal{O}(N^{-k})$ for all $k > 0$.

Proof. (i) We can suppose that x and y are monomials, since the result will follow by linearity. We will give a proof for the case of the π_+ representations, the case π_- being similar, with minor changes.

We have $\widehat{\pi}_+(y) = (\widehat{\pi}_+ a_{\beta_1})^{|\beta_1|} (\widehat{\pi}_+ b)^{\beta_2} (\widehat{\pi}_+ b^*)^{\beta_3}$. A computation gives

$$\widehat{\pi}_+(y) \varepsilon_m \otimes \varepsilon_{2j} = (-1)^{\beta_2+\beta_3} q^{(2j-m)(\beta_2+\beta_3)} q_{2j-m,|\beta_1|}^{\uparrow\beta_1} \varepsilon_{m-\beta_3+\beta_2} \otimes \varepsilon_{2j-\beta_3+\beta_2+\beta_1}$$

and with the notation $t_{2j,m} := \langle \varepsilon_m \otimes \varepsilon_{2j}, \pi'_\pm(x)\widehat{\pi}_\pm(y) \varepsilon_m \otimes \varepsilon_{2j} \rangle$ and $T_{2j} := \sum_{m=0}^{2j} t_{2j,m}$, we get

$$\begin{aligned}t_{2j,m} &= (-1)^{\beta_2+\beta_3} q^{(2j-m)(\beta_2+\beta_3)} q_{2j-m,|\beta_1|}^{\uparrow\beta_1} q_{m-\beta_3+\beta_2,|\alpha_1|}^{\uparrow\alpha_1} q^{(m+\beta_2-\beta_3)(\alpha_2+\alpha_3)} \delta_{\alpha_1+\beta_2-\beta_3,0} \\ &\quad \times \delta_{-\alpha_3+\alpha_2+\beta_1,0} \\ &= (-1)^{\alpha_1} q^{(2j-m)(\beta_2+\beta_3)+(m-\alpha_1)(\alpha_2+\alpha_3)} q_{2j-m,|\beta_1|}^{\uparrow\beta_1} q_{m-\alpha_1,|\alpha_1|}^{\uparrow\alpha_1} \delta_{\alpha_1+\beta_2-\beta_3,0} \delta_{\alpha_2-\alpha_3+\beta_1,0} \\ &=: f_{\alpha,\beta} q^{2j\lambda} t'_{2j,m} =: f_{\alpha,\beta} q^{2j\kappa} t''_{2j,2j-m}\end{aligned}$$

where

$$t'_{2j,m} := q^{m(\kappa-\lambda)} q_{2j-m,|\beta_1|}^{\uparrow\beta_1} q_{m-\alpha_1,|\alpha_1|}^{\uparrow\alpha_1}, \quad (3.33)$$

$$t''_{2j,m} := q^{m(\lambda-\kappa)} q_{m,|\beta_1|}^{\uparrow\beta_1} q_{2j-m-\alpha_1,|\alpha_1|}^{\uparrow\alpha_1}, \quad (3.34)$$

with $\lambda := \beta_2 + \beta_3 \geq 0$ and $\kappa := \alpha_2 + \alpha_3 \geq 0$. We will now prove that if $\lambda \neq \kappa$, then (T_{2j}) is a convergent sequence. Suppose $\kappa > \lambda$. Let us note $U'_{2j} := \sum_{m=0}^{2j} t'_{2j,m}$. Since the $t'_{2j,m}$ are positive and $t'_{2j+1,m} \geq t'_{2j,m}$ for all j, m , U'_{2j} is an increasing real sequence. The estimate

$$U'_{2j} \leq \sum_{m=0}^{2j} q^{m(\kappa-\lambda)} \leq \frac{1}{1-q^{\kappa-\lambda}} < \infty$$

proves then that U'_{2j} is a convergent sequence. With $T_{2j} = f_{\alpha,\beta} q^{2j\lambda} U'_{2j}$, we obtain our result.

Suppose now that $\lambda > \kappa$. Let us note $U''_{2j} := \sum_{m=0}^{2j} t''_{2j,m}$. Since the $t''_{2j,m}$ are positive and $t''_{2j+1,m} \geq t''_{2j,m}$ for all j, m , U''_{2j} is an increasing real sequence. The estimate

$$U''_{2j} \leq \sum_{m=0}^{2j} q^{m(\lambda-\kappa)} \leq \frac{1}{1-q^{\lambda-\kappa}} < \infty$$

proves then that U''_{2j} is a convergent sequence. With $T_{2j} = f_{\alpha,\beta} q^{2j\kappa} U''_{2j}$, we have again our result. Moreover, note that if λ and κ are both different from zero, the limit of (T_{2j}) is zero and more precisely,

$$T_{2j} = \mathcal{O}(q^{2j\lambda}) \text{ if } \kappa > \lambda > 0, \quad (3.35)$$

$$T_{2j} = \mathcal{O}(q^{2j\kappa}) \text{ if } \lambda > \kappa > 0. \quad (3.36)$$

Suppose now that $\lambda = \kappa \neq 0$. In that case, (T_{2j}) also converges rapidly to zero. Indeed, let us fix $q < \varepsilon < 1$. we have $\varepsilon^{-2j\lambda} T_{2j} = \sum_{m=0}^{2j} c_m d_{2j-m} = c * d(2j)$ where $c_m := f_{\alpha,\beta} (q/\varepsilon)^{\lambda m} q_{m-\alpha_1, |\alpha_1|}^{\uparrow_{\alpha_1}}$ and $d_m := (q/\varepsilon)^{\lambda m} q_{m, |\beta_1|}^{\uparrow_{\beta_1}}$. Since both $\sum_m c_m$ and $\sum_m d_m$ are absolutely convergent series, their Cauchy product $\sum_{2j} \varepsilon^{-2j\lambda} T_{2j}$ is convergent. In particular, $\lim_{j \rightarrow \infty} \varepsilon^{-2j\lambda} T_{2j} = 0$, and

$$T_{2j} = \mathcal{O}(\varepsilon^{2j\lambda}). \quad (3.37)$$

Finally, T_{2j} has a finite limit in all cases except possibly when $\lambda = \kappa = 0$, which is the case when $\alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = 0$. In that case, $t_{2j,m} = 1$.

A straightforward computation gives $\tau_1(\pi_{\pm}(x)) \tau_1(\pi_{\pm}(y)) = \delta_{\alpha_1,0} \delta_{\beta_1,0} \delta_{\alpha_2,0} \delta_{\beta_2,0} \delta_{\alpha_3,0} \delta_{\beta_3,0}$.

Thus,

$$U_{2j} = T_{2j} - (2j+1) \delta_{\alpha_1,0} \delta_{\beta_1,0} \delta_{\alpha_2,0} \delta_{\beta_2,0} \delta_{\alpha_3,0} \delta_{\beta_3,0}$$

has always a finite limit when $j \rightarrow \infty$.

(ii) The result is clear if $\lambda = \kappa = 0$ (in that case $U_N = \tau_0 = 0$). Suppose λ or κ is not zero. In that case $U_{2j} = T_{2j}$. By (3.36), (3.35) and (3.37), we see that if $\lambda > \kappa > 0$ or $\kappa > \lambda > 0$ or $\kappa = \lambda$, (T_{2j}) converges to 0 with a rate in $\mathcal{O}(\varepsilon^{2j\alpha})$ where $\alpha > 0$ and $q \leq \varepsilon < 1$. Thus, it only remains to check the cases $(\kappa > 0, \lambda = 0)$ and $(\kappa = 0, \lambda > 0)$. In the first one, we get from (3.33), $U_{2j} = f_{\alpha,\beta} \sum_{m=0}^{2j} q^{m\kappa} q_{2j-m, |\beta_1|}^{\uparrow_{\beta_1}}$. If $\beta_1 = 0$, we are done.

Suppose $\beta_1 > 0$. We have $q_{2j-m, |\beta_1|}^{\uparrow_{\beta_1}} = \sum_{p=0}^{\infty} l_p q^{r_p} q^{2|p|_1(2j-m)}$ where $p = (p_1, \dots, p_{\beta_1})$ and $l_p = (-1)^{|p|_1} \binom{\frac{1}{2}}{p}$, $r_p := 2p_1 + \dots + 2\beta_1 p_{\beta_1}$. Thus, cutting the sum in two, we get, noting $L_{2j} := f_{\alpha,\beta} \sum_{m=0}^{2j} q^{m\kappa}$,

$$U_{2j} - L_{2j} = f_{\alpha,\beta} \sum_{|p|_1 > \kappa/2} l_p q^{r_p} \frac{q^{4|p|_1 j} - q^{(2j+1)\kappa - 2|p|_1}}{1 - q^{\kappa - 2|p|_1}} + f_{\alpha,\beta} \sum_{0 \neq |p|_1 \leq \kappa/2} l_p q^{r_p} q^{4|p|_1 j} \sum_{m=0}^{2j} q^{m(\kappa - 2|p|_1)}.$$

Since $\sum_{0 \neq |p|_1 \leq \kappa/2} l_p q^{r_p} q^{4|p|_1 j} \sum_{m=0}^{2j} q^{m(\kappa - 2|p|_1)}$ is in $\mathcal{O}_{j \rightarrow \infty}(jq^{4j})$, we have, modulo a rapidly decreasing sequence,

$$U_{2j} - L_{2j} \sim f_{\alpha,\beta} \sum_{|p|_1 > \kappa/2} l_p q^{r_p} \frac{q^{4|p|_1 j} - q^{(2j+1)\kappa - 2|p|_1}}{1 - q^{\kappa - 2|p|_1}} =: f_{\alpha,\beta} q^{2\kappa j} V_{2j}$$

with

$$V_{2j} = \sum_{|p|_1 > \kappa/2} l_p q^{r_p} \frac{1 - q^{(2|p|_1 - \kappa)(2j+1)}}{1 - q^{2|p|_1 - \kappa}} = \sum_{|p|_1 > \kappa/2} \sum_{m=0}^{2j} l_p q^{r_p} q^{(2|p|_1 - \kappa)m}.$$

The family $v_{m,p} := (l_p q^{r_p} q^{(2|p|_1 - \kappa)m})_{(p,m) \in I}$, where $I = \{(p,m) \in \mathbb{N}^{\beta_1} \times \mathbb{N} : |p|_1 > \kappa/2\}$ is (absolutely) summable. Indeed $|v_{m,p}| \leq |l_p| q^{r_p} q^m$ so $|v_{m,p}|$ is summable as the product of two summable families. As a consequence, $\lim_{j \rightarrow \infty} V_{2j}$ exists and is finite, which proves that $(q^{2\kappa j} V_{2j})$, and thus $(U_{2j} - L_{2j})$ converge rapidly to 0.

Suppose now that $\beta_1 < 0$. In that case, $q_{2j-m,|\beta_1|}^{\uparrow\beta_1} = q_{2j-m,|\beta_1|}^{\downarrow} = q_{2j-(m+|\beta_1|),|\beta_1|}^{\uparrow}$ and by (3.33), we get $U_{2j} = f_{\alpha,\beta} \sum_{m=0}^{2j} q^{m\kappa} q_{2j-(m+|\beta_1|),|\beta_1|}^{\uparrow} = f_{\alpha,\beta} q^{-|\beta_1|\kappa} \sum_{m=|\beta_1|}^{2j+|\beta_1|} q^{m\kappa} q_{2j-m,|\beta_1|}^{\uparrow}$, so the same arguments as in case $\beta_1 > 0$ apply here, the summation on m simply shifted of $|\beta_1|$.

The same proof can be applied for the other case ($\kappa = 0, \lambda > 0$). This time, we only need to use (3.34) instead of (3.33) and the preceding arguments follow by replacing κ by λ and β_1 by α_1 . \square

Remark 3.4.8. *Contrary to the preceding τ_0 , the new functional contains the shift information. In particular, it filters the parts of nonzero degree.*

If $T \in X\hat{X}$, $\tilde{\rho}(T) \in \pi_+(\mathcal{A})\hat{\pi}_+(\mathcal{A}) \otimes \pi_-(\mathcal{A})\hat{\pi}_-(\mathcal{A})$.

For notational convenience, we define τ_1 on $\pi'_\pm(\mathcal{A})\hat{\pi}_\pm(\mathcal{A})$ as

$$\tau_1(\pi'_\pm(x)\hat{\pi}_\pm(y)) := \tau_1(\pi_\pm(x)) \tau_1(\pi_\pm(y)).$$

In the following, the symbol \sim_e means equals modulo a entire function.

Theorem 3.4.9. *Let $T \in X\hat{X}$. Then*

- (i) $\zeta_D^T(s) \sim_e 2(\tau_1 \otimes \tau_1)(\tilde{\rho}(T)) \zeta(s-2) + 2(\tau_0 \otimes \tau_1 + \tau_1 \otimes \tau_0)(\tilde{\rho}(T)) \zeta(s-1) + 2(\tau_0 \otimes \tau_0 - \frac{1}{2}\tau_1 \otimes \tau_1)(\tilde{\rho}(T)) \zeta(s),$
- (ii) $\oint T|\mathcal{D}|^{-3} = 2(\tau_1 \otimes \tau_1)(\tilde{\rho}(T)),$
- (iii) $\oint T|\mathcal{D}|^{-2} = 2(\tau_0 \otimes \tau_1 + \tau_1 \otimes \tau_0)(\tilde{\rho}(T)),$
- (iv) $\oint T|\mathcal{D}|^{-1} = 2(\tau_0 \otimes \tau_0 - \frac{1}{2}\tau_1 \otimes \tau_1)(\tilde{\rho}(T)).$

Proof. (i) Since $T \in X\hat{X}$, $\tilde{\rho}(T)$ is a linear combination of terms like $\pi'_+(x)\hat{\pi}_+(y) \otimes \pi'_-(z)\hat{\pi}_-(t)$, where $x, y, z, t \in \mathcal{A}$. Such a term is noted in the following $T_+ \otimes T_-$. Linear combination of these term is implicit. With the shorthand $T_{c_1, \dots, c_p} := \langle \varepsilon_{c_1} \otimes \dots \otimes \varepsilon_{c_p}, T \varepsilon_{c_1} \otimes \dots \otimes \varepsilon_{c_p} \rangle$, recalling that

$v_{m,l}^{j,\downarrow}$ is 0 when $j = 0$, or $l \geq 2j$, we get

$$\begin{aligned} \zeta_{\mathcal{D}}^T(s) &= \sum_{2j=0}^{\infty} \sum_{m=0}^{2j} \sum_{l=0}^{2j+1} \langle \begin{pmatrix} v_{m,l}^{j,\uparrow} \\ 0 \end{pmatrix}, S\tilde{\rho}(T)I \begin{pmatrix} v_{m,l}^{j,\uparrow} \\ 0 \end{pmatrix} \rangle d_{j+}^{-s} + \langle \begin{pmatrix} 0 \\ v_{m,l}^{j,\downarrow} \end{pmatrix}, S\tilde{\rho}(T)I \begin{pmatrix} 0 \\ v_{m,l}^{j,\downarrow} \end{pmatrix} \rangle d_j^{-s} \\ &= \sum_{2j=0}^{\infty} \sum_{m=0}^{2j} \sum_{l=0}^{2j+1} \tilde{\rho}(T)_{m,2j,l,2j,\uparrow} d_{j+}^{-s} + \sum_{2j=1}^{\infty} \sum_{m=0}^{2j} \sum_{l=0}^{2j-1} \tilde{\rho}(T)_{m,2j,l,2j,\downarrow} d_j^{-s} \\ &= \sum_{2j=0}^{\infty} (\text{Tr}_{2j}(T_+) \text{Tr}_{2j+1}^{\uparrow}(T_-) + \text{Tr}_{2j+1}(T_+) \text{Tr}_{2j}^{\downarrow}(T_-)) d_{j+}^{-s}. \end{aligned}$$

By Lemma 3.4.7 (ii), for all $k > 0$,

$$\begin{aligned} \text{Tr}_{2j}(T_{\pm}) &= (2j + \tfrac{3}{2})\tau_1(T_{\pm}) + \tau_0(T_{\pm}) - \tfrac{1}{2}\tau_1(T_{\pm}) + \mathcal{O}((2j)^{-k}), \\ \text{Tr}_{2j+1}(T_{\pm}) &= (2j + \tfrac{3}{2})\tau_1(T_{\pm}) + \tau_0(T_{\pm}) + \tfrac{1}{2}\tau_1(T_{\pm}) + \mathcal{O}((2j)^{-k}). \end{aligned}$$

The result follows by noting that the difference of the Hurwitz zeta function $\zeta(s, \frac{3}{2})$ and Riemann zeta function $\zeta(s)$ is an entire function.

(ii, iii, iv) are direct consequences of (i). \square

3.4.5 The smooth algebra $C^\infty(SU_q(2))$

In [31, 138], the smooth algebra $C^\infty(SU_q(2))$ is defined by pulling back the smooth structure $C^\infty(D_{q^\pm}^2)$ into the C^* -algebra generated by \mathcal{A} , through the morphism ρ and the application λ (the compression which gives an operator on \mathcal{H} from an operator on $l^2(\mathbb{N}) \otimes l^2(\mathbb{N}) \otimes l^2(\mathbb{Z}) \otimes \mathbb{C}^2$). The important point is that with [31, Lemma 2, p. 69], this algebra is stable by holomorphic calculus. By defining $\rho := \tilde{\rho} \circ c$ and $\lambda(\cdot) := S(\cdot)I$, the same lemma (with same notation) can be applied to our setting, with $c := \pi(x) \mapsto \underline{\pi}(x)$ and

$$\mathcal{C} := C^\infty(D_{q^+}^2) \otimes C^\infty(S^1) \otimes C^\infty(D_{q^+}^2) \otimes C^\infty(S^1) \otimes \mathcal{M}_2(\mathbb{C})$$

as algebra stable by holomorphic calculus containing the image of $\tilde{\rho}$. Here, we use Schwartz sequences to define the smooth structures. We finally obtain $C^\infty(SU_q(2))$ with real structure as a subalgebra stable by holomorphic calculus of the C^* -algebra generated by $\pi(\mathcal{A}) \cup J\pi(\mathcal{A})J^{-1}$ and containing $\pi(\mathcal{A}) \cup J\pi(\mathcal{A})J^{-1}$.

Corollary 3.4.10. *The dimension spectrum of the real spectral triple $(C^\infty(SU_q(2)), \mathcal{H}, D)$ is simple and given by $\{1, 2, 3\}$. Its KO -dimension is 3.*

Proof. Since F commutes with $\underline{\pi}(\mathcal{A})$, the pseudodifferential operators of order 0 (without the real structure) are exactly (modulo $OP^{-\infty}$) the operators in $\mathcal{B} + \mathcal{B}F$. From Theorem 3.3.4 we see that the dimension spectrum of $SU_q(2)$ without taking into account the reality operator J is $\{1, 2, 3\}$, in other words, the possible poles of $\zeta_{\mathcal{D}}^b : s \mapsto \text{Tr}(bF^\varepsilon|\mathcal{D}|^{-s})$ (with $\varepsilon \in \{0, 1\}$, $b \in \mathcal{B}$) are in $\{1, 2, 3\}$. Theorem 3.4.9 (i) shows that the possible poles are still $\{1, 2, 3\}$ when we take into account the real structure of $SU_q(2)$, that is to say, when \mathcal{B} is enlarged to $\mathcal{B}J\mathcal{B}J^{-1}$. Indeed, any element of $\mathcal{B}J\mathcal{B}J^{-1}$ is in $X\hat{X}$ and it is clear from the preceding proof that adding F in the previous zeta function do not add any pole to $\{1, 2, 3\}$.

All arguments go true from the polynomial algebra $\mathcal{A}(SU_q(2))$ to the smooth pre- C^* -algebra $C^\infty(SU_q(2))$.

KO-dimension refers just to $J^2 = -1$ and $\mathcal{D}J = J\mathcal{D}$ since there is no chirality because spectral dimension is 3. \square

3.4.6 Noncommutative integrals with reality operator and one-forms on $SU_q(2)$

The goal of this section is to obtain the following suppression of J :

Theorem 3.4.11. *Let A and B be δ -one-forms. Then*

- (i) $\oint AJBJ^{-1}|\mathcal{D}|^{-3} = \frac{1}{2} \oint A|\mathcal{D}|^{-3} \overline{\oint B|\mathcal{D}|^{-3}},$
- (ii) $\oint AJBJ^{-1}|\mathcal{D}|^{-2} = \frac{1}{2} \oint A|\mathcal{D}|^{-2} \overline{\oint B|\mathcal{D}|^{-3}} + \frac{1}{2} \oint A|\mathcal{D}|^{-3} \overline{\oint B|\mathcal{D}|^{-2}},$
- (iii) $\oint A^2JBK^{-1}|\mathcal{D}|^{-3} = \frac{1}{2} \oint A^2|\mathcal{D}|^{-3} \overline{\oint B|\mathcal{D}|^{-3}},$
- (iv) $\oint \delta(A)A|\mathcal{D}|^{-3} = \oint \delta(A)JAJ^{-1}|\mathcal{D}|^{-3} = 0.$

We gather at the beginning of this section the main notations for technical lemmas which will follow.

For any pair $(k, p) \in \mathbb{N}^3 \times \mathbb{N}^3$ such that $k_i \leq |\alpha_i|$, $p_i \leq |\beta_i|$, where $\alpha, \beta \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N}$, we define

$$\begin{aligned} v_{k,p} &:= g(p) \binom{|\alpha_1|}{k_1}_{q^{2\eta_{\alpha_1}}} \binom{\alpha_2}{k_2} \binom{\alpha_3}{k_3} \binom{|\beta_1|}{p_1}_{q^{2\eta_{\beta_1}}} \binom{\beta_2}{p_2} \binom{\beta_3}{p_3} (-1)^{k_1+p_1+\alpha_2+\alpha_3+\beta_2+\beta_3} q^{\sigma_{k,p}}, \\ h_{k,p} &:= \alpha_1 + \alpha_2 - \alpha_3 - 2(\eta_{\alpha_1}k_1 + k_2 - k_3) + g(p), \\ g(p) &:= \beta_1 + \beta_2 - \beta_3 - 2(\eta_{\beta_1}p_1 + p_2 - p_3), \\ \sigma_{k,p} &:= k_1 + p_1 + \sigma_{k,p}^t + \sigma_{k,p}^u, \\ \sigma_{k,p}^t &:= k_1\widehat{k}_2 - \widehat{k}_3(k_1 + k_2) + \eta_{\beta_1}\widehat{p}_1|k|_1 + \widehat{p}_2(|k|_1 + p_1) - \widehat{p}_3(|k|_1 + p_1 + p_2), \\ \sigma_{k,p}^u &:= (k_3 + \eta_{\beta_1}\widehat{p}_1 - p_2 + p_3)(k_1 + \widehat{k}_2 + \widehat{k}_3) - k_2(k_1 + \widehat{k}_2) + (p_1 + \widehat{p}_2)(-p_2 + p_3) + \widehat{p}_3p_3, \\ t_{k,p} &= a_{\alpha_1}^{\widehat{k}_1} a^{\widehat{k}_2} a^{*\widehat{k}_3} a_{\beta_1}^{\widehat{p}_1} a^{\widehat{p}_2} a^{*\widehat{p}_3} b^{|k|_1+|p|_1}, \\ u_{k,p} &= a_{\alpha_1}^{\widehat{k}_1} a^{*\widehat{k}_2} a^{k_3} a_{\beta_1}^{\widehat{p}_1} a^{*p_2} a^{p_3} b^{|\widehat{k}|_1+|\widehat{p}|_1}. \end{aligned}$$

where we used the notation

$$\widehat{k}_i := |\alpha_i| - k_i, \quad \widehat{p}_i := |\beta_i| - p_i,$$

so $0 \leq \widehat{k}_i \leq |\alpha_i|$, $0 \leq \widehat{p}_i \leq |\beta_i|$. We will also use the shorthand $\widetilde{k} := (k_1, \widehat{k}_2, \widehat{k}_3)$.

For $\beta_1 \in \mathbb{Z}$ and $j \in \mathbb{N}$, we define

$$\begin{aligned} w_1(\beta_1, j) &:= \sum_{n=0}^{\infty} (q^{2jn} (q_{n,|\beta_1|}^{\uparrow \beta_1})^2 - \delta_{j,0}), \\ w_{\beta}^{\alpha} &:= 2\beta_1 q^{\beta_1(2\alpha_3+\beta_3-\beta_2)} w_1(\beta_1, \alpha_3 + \beta_3). \end{aligned}$$

We introduce the following notations:

$$\begin{aligned}
q_{k,p,n}^+ &:= q^{n(|\tilde{k}|_1+|\tilde{p}|_1)} q_{n+r_{k,p}^+-\eta_{\alpha_1}\widehat{k}_1,\widehat{k}_1}^{\uparrow_{\alpha_1}} q_{n-\widehat{k}_3+\eta_{\beta_1}\widehat{p}_1+\widehat{p}_2-\widehat{p}_3,\widehat{k}_2}^{\uparrow} q_{n+\eta_{\beta_1}\widehat{p}_1+\widehat{p}_2-\widehat{p}_3,\widehat{k}_3}^{\downarrow} q_{n+\widehat{p}_2-\widehat{p}_3,\widehat{p}_1}^{\uparrow_{\beta_1}} q_{n-\widehat{p}_3,\widehat{p}_2}^{\uparrow} q_{n,\widehat{p}_3}^{\downarrow}, \\
q_{k,p,n}^- &:= q^{n(|\tilde{k}|_1+|\tilde{p}|_1)} q_{n+r_{k,p}^--\eta_{\alpha_1}\widehat{k}_1,\widehat{k}_1}^{\uparrow_{\alpha_1}} q_{n+k_3+\eta_{\beta_1}\widehat{p}_1-p_2+p_3,\widehat{k}_2}^{\downarrow} q_{n+\eta_{\beta_1}\widehat{p}_1-p_2+p_3,\widehat{k}_3}^{\uparrow} q_{n-p_2+p_3,\widehat{p}_1}^{\uparrow_{\beta_1}} q_{n+p_3,\widehat{p}_2}^{\downarrow} q_{n,p_3}^{\uparrow} \\
&\quad \times (-1)^{|\tilde{k}|_1+|\tilde{p}|_1}, \\
r_{k,p}^+ &:= \eta_{\alpha_1}\widehat{k}_1 + \widehat{k}_2 - \widehat{k}_3 + \eta_{\beta_1}\widehat{p}_1 + \widehat{p}_2 - \widehat{p}_3, \\
r_{k,p}^- &:= \eta_{\alpha_1}\widehat{k}_1 - k_2 + k_3 + \eta_{\beta_1}\widehat{p}_1 - p_2 + p_3.
\end{aligned}$$

Thus, $\pi_+(t_{k,p})\varepsilon_n = q_{k,p,n}^+\varepsilon_{n+r_{k,p}^+}$ and $\pi_-(u_{k,p})\varepsilon_n = q_{k,p,n}^-\varepsilon_{n+r_{k,p}^-}$.

Lemma 3.4.12. *We have*

$$r((M_\beta^\alpha)^\circ) = \sum_{k,p} \delta_{h_{k,p},0} v_{k,p} \pi_+(t_{k,p}) \otimes \pi_-(u_{k,p})$$

where the summation is done on k_i, p_i in \mathbb{N} such that $k_i \leq |\alpha_i|, p_i \leq |\beta_i|$ for $i \in \{1, 2, 3\}$.

Proof. Since $\pi(m^\alpha) = (a_+ + a_-)^{\alpha_1} (b_+ + b_-)^{\alpha_2} (b_+^* + b_-^*)^{\alpha_3}$, with $v_k := \binom{|\alpha_1|}{k_1}_{q^{2\eta_{\alpha_1}}} \binom{\alpha_2}{k_2} \binom{\alpha_3}{k_3}$,

$$\pi(m^\alpha) = \sum_k v_k c_k \text{ where } c_k := a_{+,\alpha_1}^{|\alpha_1|-k_1} a_{-,\alpha_1}^{k_1} b_+^{\alpha_2-k_2} b_-^{k_2} b_+^*{}^{\alpha_3-k_3} b_-^*{}^{k_3}.$$

By Lemma 3.3.2 (iii) we see that $\delta(\pi(m^\beta)) = \sum_p w_p d_p$ where we introduce

$$w_p := \binom{|\beta_1|}{p_1}_{q^{2\eta_{\beta_1}}} \binom{\beta_2}{p_2} \binom{\beta_3}{p_3} \text{ and } d_p := g(p) a_{+,\beta_1}^{|\beta_1|-p_1} a_{-,\beta_1}^{p_1} b_+^{\beta_2-p_2} b_-^{p_2} b_+^*{}^{\beta_3-p_3} b_-^*{}^{p_3}.$$

As a consequence, $(M_\beta^\alpha)^\circ = \sum_{k,p} \delta_{h(k,p),0} g(p) v_k w_p c_{k,p}$ where

$$c_{k,p} = a_{+,\alpha_1}^{\widehat{k}_1} a_{-,\alpha_1}^{k_1} b_+^{\widehat{k}_2} b_-^{k_2} b_+^*{}^{\widehat{k}_3} b_-^*{}^{k_3} a_{+,\beta_1}^{\widehat{p}_1} a_{-,\beta_1}^{p_1} b_+^{\widehat{p}_2} b_-^{p_2} b_+^*{}^{\widehat{p}_3} b_-^*{}^{p_3} \quad (3.38)$$

With (3.38), we get $r(c_{k,p}) = (-1)^{k_1+p_1+\alpha_2+\alpha_3+\beta_2+\beta_3} q^{k_1+p_1} \pi_+(t'_{k,p}) \otimes \pi_-(u'_{k,p})$ where

$$\begin{aligned}
t'_{k,p} &= a_{\alpha_1}^{\widehat{k}_1} b^{k_1} a^{\widehat{k}_2} b^{k_2} a^{*\widehat{k}_3} b^{k_3} a_{\beta_1}^{\widehat{p}_1} b^{p_1} a^{\widehat{p}_2} b^{p_2} a^{*\widehat{p}_3} b^{p_3}, \\
u'_{k,p} &= a_{\alpha_1}^{\widehat{k}_1} b^{k_1} b^{\widehat{k}_2} a^{*k_2} b^{\widehat{k}_3} a^{k_3} a_{\beta_1}^{\widehat{p}_1} b^{p_1} b^{\widehat{p}_2} a^{*p_2} b^{\widehat{p}_3} a^{p_3}.
\end{aligned}$$

A recursive use of relation $ba_j = q^{\eta_j} a_j b$ yields the result. \square

Lemma 3.4.13. *We have*

$$(i) (\tau_1 \otimes \tau_1)(r(M_\beta^\alpha)^\circ) = \beta_1 \delta_{\alpha_1, -\beta_1} \delta_{\alpha_2, 0} \delta_{\alpha_3, 0} \delta_{\beta_2, 0} \delta_{\beta_3, 0}.$$

$$(ii) (\tau_1 \otimes \tau_0 + \tau_0 \otimes \tau_1)(r(M_\beta^\alpha)^\circ) = \delta_{\alpha_1, -\beta_1} \delta_{\alpha_2+\beta_2, \alpha_3+\beta_3} w_\beta^\alpha.$$

In particular, if A is a δ -one-form, we have

$$\begin{aligned}
\oint A |\mathcal{D}|^{-3} &= 2\beta_1 A_{-\beta_1 00}^{\beta_1 00}, \\
\oint A |\mathcal{D}|^{-2} &= 2w_\beta^\alpha B(A)_\alpha^\beta.
\end{aligned}$$

where we implicitly summed on all α, β indices.

Proof. See Appendix A. \square

With notations of Lemma 3.4.12, it is direct to check that for given $\bar{\alpha} = (\alpha, \alpha', \dots, \alpha^{(n-1)})$ and $\bar{\beta} = (\beta, \beta', \dots, \beta^{(n-1)})$,

$$r((M_{\bar{\beta}}^{\bar{\alpha}})^{\circ}) = \sum_{K,P} \delta_{h_{K,P},0} v_{K,P} \pi_+(t_{K,P}) \otimes \pi_-(u_{K,P}) \quad (3.39)$$

where $K = (k, k', \dots, k^{(n-1)})$, $P = (p, p', \dots, p^{(n-1)})$ with $0 \leq k_i^{(j)} \leq |\alpha_i^{(j)}|$, $0 \leq p_i^{(j)} \leq |\beta_i^{(j)}|$,

$$\begin{aligned} t_{K,P} &:= t_{k,p} t_{k',p'} \cdots t_{k^{(n-1)},p^{(n-1)}} , & u_{K,P} &:= u_{k,p} u_{k',p'} \cdots u_{k^{(n-1)},p^{(n-1)}} , \\ v_{K,P} &:= v_{k,p} v_{k',p'} \cdots v_{k^{(n-1)},p^{(n-1)}} , & h_{K,P} &:= h_{k,p} + h_{k',p'} + \cdots h_{k^{(n-1)},p^{(n-1)}} . \end{aligned}$$

In the following, we will use the shorthands $A_i := \alpha_i + \alpha'_i + \cdots + \alpha_i^{(n-1)}$, $B_i := \beta_i + \beta'_i + \cdots + \beta_i^{(n-1)}$.

In the case $n = 2$, we also note $r_{K,P}^{\pm} := r_{k,p}^{\pm} + r_{k',p'}^{\pm}$ and $q_{K,P,n}^{\pm} := q_{k',p',n}^{\pm} q_{k,p,n+r_{k',p'}^{\pm}}^{\pm}$.

Thus, we have $\pi_+(t_{K,P}) \varepsilon_m = q_{K,P,m}^+ \varepsilon_{m+r_{K,P}^+}$ and $\pi_-(u_{K,P}) \varepsilon_m = q_{K,P,n}^- \varepsilon_{m+r_{K,P}^-}$.

We also introduce, still for $n = 2$,

$$\begin{aligned} v_{\beta_1, \alpha'_1, \beta'_1}(l, j) &:= \sum_{n=0}^{\infty} (q^{l+2nj} q_{n+\beta'_1+\alpha'_1+\beta_1, |\beta'_1+\alpha'_1+\beta_1|}^{\uparrow_{-\beta'_1-\alpha'_1-\beta_1}} q_{n+\beta'_1+\alpha'_1, |\beta_1|}^{\uparrow_{\beta_1}} q_{n+\beta'_1, |\alpha'_1|}^{\uparrow_{\alpha'_1}} q_{n, |\beta'_1|}^{\uparrow_{\beta'_1}} - \delta_{j,0}), \\ V_{\bar{\beta}}^{\bar{\alpha}} &:= 2[\beta_1 \beta'_1 + (\beta_2 - \beta_3)(\beta'_2 - \beta'_3)] q^{2\beta_1(\alpha_2+\alpha_3)+2\beta'_1(\alpha'_2+\alpha'_3)} \\ &\quad \times v_{\beta_1, \alpha'_1, \beta'_1}((\alpha_2 + \beta_2 + \alpha_3 + \beta_3)(\alpha'_1 + \beta'_1), A_3 + B_3). \end{aligned}$$

Lemma 3.4.14. *We have*

- (i) $(\tau_1 \otimes \tau_1) (r(M_{\bar{\beta}}^{\alpha} M_{\bar{\beta}'}^{\alpha'})^{\circ}) = \beta_1 \beta'_1 \delta_{A_1, -B_1} \delta_{A_2, 0} \delta_{A_3, 0} \delta_{B_2, 0} \delta_{B_3, 0}.$
- (ii) $(\tau_1 \otimes \tau_0 + \tau_0 \otimes \tau_1) (r(M_{\bar{\beta}}^{\alpha} M_{\bar{\beta}'}^{\alpha'})^{\circ}) = \delta_{A_2+B_2, A_3+B_3} \delta_{A_1, -B_1} V_{\bar{\beta}}^{\bar{\alpha}}.$
- (iii) $(\tau_1 \otimes \tau_1) (r(M_{\bar{\beta}}^{\alpha} M_{\bar{\beta}'}^{\alpha'} M_{\bar{\beta}''}^{\alpha''})^0) = \beta_1 \beta'_1 \beta''_1 \delta_{A_1, -B_1} \delta_{A_2, 0} \delta_{A_3, 0} \delta_{B_2, 0} \delta_{B_3, 0}.$
- (iv) $(\tau_1 \otimes \tau_1) (r(\delta(M_{\bar{\beta}}^{\alpha}) M_{\bar{\beta}'}^{\alpha'})^0) = -(\alpha'_1 + \beta'_1) \beta_1 \beta'_1 \delta_{A_1, -B_1} \delta_{A_2, 0} \delta_{A_3, 0} \delta_{B_2, 0} \delta_{B_3, 0}.$
- (v) *In particular, if A is a δ -one-form,*

$$\begin{aligned} \oint A^2 |\mathcal{D}|^{-3} &= 2\beta_1 \beta'_1 B_a(A^2)_{\bar{\alpha}}^{\bar{\beta}}, \\ \oint A^2 |\mathcal{D}|^{-2} &= 2V_{\bar{\beta}}^{\bar{\alpha}} B(A^2)_{\bar{\alpha}}^{\bar{\beta}}, \\ \oint A^3 |\mathcal{D}|^{-3} &= 2\beta_1 \beta'_1 \beta''_1 B_a(A^3)_{\bar{\alpha}}^{\bar{\beta}}, \\ \oint \delta(A) A |\mathcal{D}|^{-3} &= \oint A \delta(A) |\mathcal{D}|^{-3} = 0. \end{aligned}$$

Proof. See Appendix B. \square

For a given δ -1-form A , we say that A is homogeneous of degree in a equal to $n \in \mathbb{Z}$ if it is a linear combination of $M_{\bar{\beta}}^{\alpha}$ such that $\alpha_1 + \beta_1 = n$. From Lemma 3.4.14 (iv) we get,

Corollary 3.4.15. *Let A, A' be two δ -1-forms, then*

$$\begin{aligned} \oint (A|\mathcal{D}|^{-1})^2 &= \oint A^2|\mathcal{D}|^{-2}, \\ \oint A|\mathcal{D}|^{-1}A'|\mathcal{D}|^{-1} &= \oint AA'|\mathcal{D}|^{-2} - n \oint AA'|\mathcal{D}|^{-3}, \text{ when } A' \text{ homogenous of degree } n. \end{aligned}$$

Lemma 3.4.16. *We have*

- (i) $(\tau_1 \otimes \tau_1) \tilde{\rho}(M_\beta^\alpha J M_{\beta'}^{\alpha'} J^{-1}) = \beta_1 \beta'_1 \delta_{\alpha_1, -\beta_1} \delta_{\alpha'_1, -\beta'_1} \delta_{A_2, 0} \delta_{A_3, 0} \delta_{B_2, 0} \delta_{B_3, 0}.$
- (ii) $(\tau_0 \otimes \tau_1 + \tau_1 \otimes \tau_0) \tilde{\rho}(M_\beta^\alpha J M_{\beta'}^{\alpha'} J^{-1}) = \delta_{\alpha_1, -\beta_1} \delta_{\alpha'_1, -\beta'_1} (\beta'_1 w_\beta^\alpha \delta_{\alpha_2 + \beta'_2 + \alpha'_3 + \beta'_3, 0} \delta_{\alpha_2 + \beta_2, \alpha_3 + \beta_3} + \beta_1 w_{\beta'}^{\alpha'} \delta_{\alpha_2 + \beta_2 + \alpha_3 + \beta_3, 0} \delta_{\alpha'_2 + \beta'_2, \alpha'_3 + \beta'_3}).$
- (iii) $(\tau_1 \otimes \tau_1) \tilde{\rho}(M_\beta^\alpha M_{\beta'}^{\alpha'} J M_{\beta''}^{\alpha''} J^{-1}) = \beta_1 \beta'_1 \beta''_1 \delta_{\alpha_1 + \alpha'_1, -\beta_1 - \beta'_1} \delta_{\alpha''_1, -\beta''_1} \delta_{A_2, 0} \delta_{A_3, 0} \delta_{B_2, 0} \delta_{B_3, 0}.$
- (iv) $(\tau_1 \otimes \tau_1) \tilde{\rho}(\delta(M_\beta^\alpha) J M_{\beta'}^{\alpha'} J^{-1}) = -(\alpha'_1 + \beta'_1) \beta_1 \beta'_1 \delta_{\alpha_1, -\beta_1} \delta_{\alpha'_1, -\beta'_1} \delta_{A_2, 0} \delta_{A_3, 0} \delta_{B_2, 0} \delta_{B_3, 0}.$
- (v) *In particular, if A and A' are δ -one forms,*

$$\begin{aligned} \oint A J A' J^{-1} |\mathcal{D}|^{-3} &= 2(\beta_1 A_{-\beta_1 00}^{\beta_1 00})(\beta_1 \bar{A}'^{\beta_1 00}_{-\beta_1 00}), \\ \oint A J A' J^{-1} |\mathcal{D}|^{-2} &= 2(\beta_1 \bar{A}'^{\beta_1 00}_{-\beta_1 00})(w_\beta^\alpha B(A)_\alpha^\beta) + 2(\beta_1 A_{-\beta_1 00}^{\beta_1 00})(w_\beta^\alpha B(\bar{A}')_\alpha^\beta), \\ \oint A^2 J A' J^{-1} |\mathcal{D}|^{-3} &= 2(\beta_1 \bar{A}'^{\beta_1 00}_{-\beta_1 00})(\beta_1 \beta'_1 B_a(A^2)_{\bar{\alpha}}^{\bar{\beta}}), \\ \oint \delta(A) J A J^{-1} &= 0. \end{aligned}$$

Proof. See Appendix C. □

Lemma 3.4.17. *Let $\beta, \beta' \in \mathbb{Z}$. Then,*

$$\lim_{2j \rightarrow \infty} \sum_{m=0}^{2j} ((q_{m, |\beta|}^{\uparrow \beta} q_{2j-m, |\beta'|}^{\uparrow \beta'})^2 - 1) = \sum_{m=0}^{\infty} ((q_{m, |\beta|}^{\uparrow \beta})^2 - 1) + \sum_{m=0}^{\infty} ((q_{m, |\beta'|}^{\uparrow \beta'})^2 - 1).$$

Proof. See Appendix D. □

Proof of Theorem 3.4.11. The result follows from Lemmas 3.4.13, 3.4.14 (v) and 3.4.16 (v). □

3.4.7 Proof of Theorem 3.4.3 and corollaries

Lemma 3.4.18. *We have on $SU_q(2)$,*

- (i) $\oint |\mathcal{D}|^{-3} = 2.$
- (ii) $\oint |\mathcal{D}|^{-2} = 0.$
- (iii) $\oint |\mathcal{D}|^{-1} = -\frac{1}{2}.$
- (iv) $\zeta_{\mathcal{D}}(0) = 0.$

Proof. (iv) We have by definition

$$\zeta_{\mathcal{D}}(s) := \text{Tr}(|\mathcal{D}|^{-s}) = \sum_{2j=0}^{\infty} \sum_{m=0}^{2j} \sum_{l=0}^{2j+1} \langle v_{m,l}^j, |\mathcal{D}|^{-s} v_{m,l}^j \rangle.$$

Since $|\mathcal{D}|^{-s} v_{m,l}^j = \begin{pmatrix} d_{j+}^{-s} & 0 \\ 0 & d_j^{-s} \end{pmatrix} v_{m,l}^j$ where $d_j := 2j + \frac{1}{2}$, we get

$$\zeta_{\mathcal{D}}(s) = \sum_{2j=0}^{\infty} (2j+1)(2j+2) d_{j+}^{-s} + \sum_{2j=1}^{\infty} (2j+1)(2j) d_j^{-s} = 2 \sum_{2j=0}^{\infty} (2j+1)(2j) d_j^{-s}.$$

With the equalities $(2j+1)(2j) = d_j^2 - \frac{1}{4}$ and $\zeta(s, \frac{1}{2}) = (2^s - 1)\zeta(s)$ (here $\zeta(s, x) := \sum_{n \in \mathbb{N}} \frac{1}{(n+x)^s}$ is the Hurwitz zeta function and $\zeta(s) := \zeta(s, 1)$ is the Riemann zeta function) we get

$$\zeta_{\mathcal{D}}(s) = 2(2^{s-2} - 1)\zeta(s-2) - \frac{1}{2}(2^s - 1)\zeta(s) \quad (3.40)$$

which entails that $\zeta_{\mathcal{D}}(0) = 0$.

(i, ii, iii) are direct consequences of equation (3.40). \square

Proof of Theorem 3.4.3. It is a consequence of Lemma 3.4.18 and Theorems 3.4.1, 3.4.11. \square

As we have seen, the computation of the noncommutative integral on $SU_q(2)$ leads to certain functions of A which filter some symmetry on the degree in a, a^*, b, b^* of the canonical decomposition. Precisely, it is the balanced features that appear and the following functions of $A^n, n \in \{1, 2, 3\}$:

$$\oint A^n |\mathcal{D}|^{-p} \quad (3.41)$$

where $1 \leq n \leq p \leq 3$. In the next section, we describe a method to compute these integrals.

Corollary 3.4.19. *Let u be a unitary in $C^\infty(SU_q(2))$ and $\gamma_u(\mathbb{A}) := \pi(u)\mathbb{A}\pi(u^*) + \pi(u)d\pi(u^*)$ be a gauge-variant of \mathbb{A} . Then the following terms of Theorem 3.4.3 are gauge invariant*

$$\oint A |\mathcal{D}|^{-3}, \quad \oint A^2 |\mathcal{D}|^{-3} - \oint A |\mathcal{D}|^{-2}, \quad -2 \oint A |\mathcal{D}|^{-1} + \oint A^2 |\mathcal{D}|^{-2} - \frac{2}{3} \oint A^3 |\mathcal{D}|^{-3}.$$

Proof. It is sufficient to remark that all terms $\oint |D_{\mathbb{A}}|^{-k}$ and $\zeta_{\mathcal{D}_{\mathbb{A}}}(0)$ in the spectral action (1.6) are gauge invariant. This can also be seen via the computation $D_{\gamma_u(\mathbb{A})} = V_u \mathcal{D} V_u^* + V_u P_0 V_u^*$ where P_0 is the projection on $\text{Ker } \mathcal{D}$ and $V_u = \pi(u)J\pi(u)J^{-1}$ and $\oint |D_{\mathbb{A}}|^{n-k} = \text{Res}_{s=n-k} \text{Tr} (|D_{\mathbb{A}}|^{n-k})$ (see Proposition 2.2.1 (iii) and Proposition 1.3.8). \square

Corollary 3.4.20. *In the case of the spectral action without the reality operator (i.e. $D_{\mathbb{A}} = \mathcal{D} + \mathbb{A}$), we get*

$$\begin{aligned} \oint |D_{\mathbb{A}}|^{-3} &= 2, & \oint |D_{\mathbb{A}}|^{-2} &= -2 \oint A |\mathcal{D}|^{-3}, & \oint |D_{\mathbb{A}}|^{-1} &= -\frac{1}{2} - \oint A |\mathcal{D}|^{-2} + \oint A^2 |\mathcal{D}|^{-3}, \\ \zeta_{D_{\mathbb{A}}}(0) &= -\oint A |\mathcal{D}|^{-1} + \frac{1}{2} \oint A^2 |\mathcal{D}|^{-2} - \frac{1}{3} \oint A^3 |\mathcal{D}|^{-3}. \end{aligned}$$

As a consequence, if \mathbb{A} is a one-form such that $\oint A |\mathcal{D}|^{-3} = 0$, then the scale invariant term of the spectral action with or without J is exactly the same modulo a global factor of 2.

3.5 Differential calculus on $SU_q(2)$ and applications

3.5.1 The sign of \mathcal{D}

There are multiple differential calculi on $SU_q(2)$, see [92, 153]. Due to [121, Theorem 3], the $3D$ and $4D_{\pm}$ differential calculi do not coincide with the one considered here: the right multiplication of one-forms by an element in the algebra \mathcal{A} is a consequence of the chosen Dirac operator which was introduced according to some equivariance properties with respect to the duality between the two Hopf algebras $SU_q(2)$ and $\mathcal{U}_q(su(2))$.

It is known that the Fredholm module associated to $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is one-summable since $[F, \pi(x)]$ is trace-class for all $x \in \mathcal{A}$. In fact, more can be said about F ¹:

Proposition 3.5.1. *Since*

$$\frac{1}{1-q^2} (\pi(a^*) d\pi(a) + q^2 \pi(b) d\pi(b^*) + q^2 \pi(a) d\pi(a^*) + q^2 \pi(b^*) d\pi(b)) = F, \quad (3.42)$$

F is a central one-form modulo $OP^{-\infty}$.

Proof. Forgetting π , this follows from

$$\begin{aligned} a^* \delta a + q^2 b \delta b^* + q^2 a \delta a^* + q^2 b^* \delta b \\ &= (a_+^* + a_-^*)(a_+ - a_-) + q^2 (b_+ + b_-)(b_-^* - b_+^*) + q^2 (a_+ + a_-)(a_-^* - a_+^*) \\ &\quad + q^2 (b_+^* + b_-^*)(b_+ - b_-) \\ &= [a_+^* a_+ - q^2 a_+ a_+^* + q^2 b_+^* b_+ - q^2 b_+ b_+^*] + R = (1 - q^2) + R \end{aligned} \quad (3.43)$$

by (3.12) where we check that the remainder R is zero:

$$\begin{aligned} R = & -[a_+^* a_- + q^2 b_+^* b_-] + [a_-^* a_+ + q^2 b_-^* b_+] - [a_-^* a_- - q^2 a_- a_-^* + q^2 b_-^* b_- - q^2 b_- b_-^*] \\ & + (q^2 a_+ a_-^* + q^2 q_-^* b_+) - (a_+^* a_- + q^2 b_+^* b_-), \end{aligned}$$

thus, applying (3.15), (3.16), (3.17), $R = +(q^2 a_+ a_-^* + q^2 q_-^* b_+) - (a_+^* a_- + q^2 b_+^* b_-) = 0$ using commutation relations (3.11).

Now, replacing δ by d in (3.43) gives (3.42) since F commute with a_{\pm} , b_{\pm} and F is central by (3.24). \square

Proposition 3.5.2. *The one-form in (3.42) is in fact exactly a function of the Dirac operator \mathcal{D} :*

$$\pi(a^*) d\pi(a) + q^2 \pi(b) d\pi(b^*) + q^2 \pi(a) d\pi(a^*) + q^2 \pi(b^*) d\pi(b) = \xi_q(\mathcal{D}) = F \xi_q(|\mathcal{D}|), \quad (3.44)$$

where $\xi_q(s) := q^{\frac{[2s]-2s}{[s+1/2][s-1/2]}}$.

Moreover, $F = \lim_{q \rightarrow 0} \xi_q(\mathcal{D})$.

Proof. First, let us observe that the one-form ω in (3.44) is invariant under the action of the $\mathcal{U}_q(su(2)) \times \mathcal{U}_q(su(2))$: $h \triangleright \omega = \epsilon(h) \omega$ for any $h \in \mathcal{U}_q(su(2)) \times \mathcal{U}_q(su(2))$. For instance, using notations of [48]

$$e \triangleright \omega = q^{\frac{1}{2}} a^* db + q^2 \left(-q^{\frac{1}{2}-1} b da^* + q^{-\frac{1}{2}} b da^* - q^{-1-\frac{1}{2}} a^* db \right) = 0 = \epsilon(e) \omega.$$

¹Note that a similar result for a different spectral triple over $SU_q(2)$ when $q = 0$ was obtained in [31, eq. (48)]

Therefore, since both the representation π as well as the operator \mathcal{D} are equivariant, the image of ω must be diagonal in the spinorial base. A tedious computation with the full spinorial representation π given in (3.6) yields

$$\begin{aligned}\langle v_{ml}^{j\uparrow}, \omega v_{ml}^{j\uparrow} \rangle &= \frac{q^{8j+8} - q^{8j+6} - (4j+3)q^{4j+6} + (8j+6)q^{4j+4} - (4j+3)q^{4j+2} - q^2 + 1}{(q^{4j+4} - 1)(q^{4j+2} - 1)} = \xi_q(2j + \tfrac{3}{2}), \\ \langle v_{ml}^{j\downarrow}, \omega v_{ml}^{j\downarrow} \rangle &= \frac{-q^{8j+4} + q^{8j+2} + (4j+1)q^{4j+4} - (8j+2)q^{4j+2} + (4j+1)q^{4j} + q^2 - 1}{(q^{4j+2} - 1)(q^{4j} - 1)} = -\xi_q(2j + \tfrac{1}{2}).\end{aligned}$$

These expressions have a clear $q = 0$ limit equal respectively to 1 and -1, so $\omega \rightarrow F$ as $q \rightarrow 0$. \square

In the $q = 1$ limit, these expressions yields identically 0, which is confirmed by the fact that all one-forms are central, it could be expressed as $d(aa^* + bb^*) = d1$.

Note that since the invariant one-form we constructed differs by $OP^{-\infty}$ from F , hence any commutator with it will be itself in $OP^{-\infty}$.

We do not know if a central form ω is automatically invariant by the action of both $U_q(su(2))$, that is: $h \triangleright \omega = \epsilon(h)\omega$.

Proposition 3.5.3. *The order one calculus up to $OP^{-\infty}$ is not universal.*

Proof. Let us take the one-form ω_F from (3.42), which gives F . Then, for any $x \in \mathcal{A}(SU_q(2))$ we have $\pi(x\omega_F - \omega_F x) = 0$. \square

Note that since $\omega_F \sim (1 - q^2)^{-1}\omega \pmod{OP^{-\infty}}$, we get $1 \sim (1 - q^2)^{-1}\xi_q(|\mathcal{D}|) \pmod{OP^{-\infty}}$.

Corollary 3.5.4. *Still modulo $OP^{-\infty}$, $1 \in \pi(\Omega_u^2(\mathcal{A}))$.*

Proof. $1 = F^2$ is by definition in $\pi(\Omega_u^2(\mathcal{A}))$. \square

In fact, one checks, using (3.12), (3.15), (3.18) that

$$q^2 da da^* - da^* da = 1 - q^2 \tag{3.45}$$

showing again that $1 \in \pi(\Omega_u^2(\mathcal{A}))$.

Similarly, using (3.11) and (3.13), (3.18), (3.19), we get still up to $OP^{-\infty}$

$$\begin{aligned}q da db &= db da, & q da db^* &= db^* da, \\ da^* db &= q db da^*, & da^* db^* &= q db^* da^* \\ db db^* &= db^* db, & da da^* + db db^* &= -1.\end{aligned} \tag{3.46}$$

The use of the last equality of (3.46) and (3.45) gives

Proposition 3.5.5. *Up to $OP^{-\infty}$, F is not a (universal) closed one-form, as*

$$da^* da + q^2 da da^* + q^2 db^* db + q^2 db db^* = -1 - q^2. \tag{3.47}$$

3.5.2 The ideal \mathcal{R}

In order to perform explicit calculations of all terms of the spectral action, we observe that each δ -one-form could be expressed in terms of $x\delta(z)y$, where z is one of the generators a, a^*, b, b^* and x, y are some elements of the algebra $\mathcal{A}(SU_q(2))$.

Then, for the computation of $\oint x dzy |\mathcal{D}|^{-1}$ we can use the trace property of the noncommutative integral to get:

$$\oint x \delta(z) y |\mathcal{D}|^{-1} = \oint y x \delta(z) |\mathcal{D}|^{-1} + \oint x \delta(z) |\mathcal{D}|^{-1} \delta(y) |\mathcal{D}|^{-1}.$$

Therefore, the problem of calculating the tadpole-like integral could be in effect reduced to the calculation of much simpler integrals: $\oint x \delta(z) |\mathcal{D}|^{-1}$ for all generators z and the integrals of higher order in $|\mathcal{D}|^{-1}$.

However, it appears that the calculations of higher-order terms simplify a lot, when we further restrict the algebra by introducing an ideal, which is *invisible* to the parts of integral at dimension 2 and 3. For instance, consider the space of pseudodifferential operators $T \in \Psi^0(\mathcal{A})$ of order less or equal to zero (see [38]), which satisfy

$$\oint T t |\mathcal{D}|^{-2} = \oint t T |\mathcal{D}|^{-2} = \oint T t |\mathcal{D}|^{-3} = \oint t T |\mathcal{D}|^{-3} = 0, \quad \forall t \in \Psi_0^0(\mathcal{A}). \quad (3.48)$$

The elements $a_-, b_- b_+, b_- b_+^*$ and their adjoints are in this space up to $OP^{-\infty}$: this is due to the fact that in Theorem 3.3.4, $\tau_1 \otimes \tau_1(r(x)) = 0$ when $r(x) \in \pi_{\pm}(\mathcal{A}) \otimes \pi_{\pm}(\mathcal{A}) \bmod OP^{-\infty}$ contains tensor products of $\pi_{\pm}(b)$ or $\pi_{\pm}(b^*)$ since these elements are in the kernel of σ .

Definition 3.5.6. Let R be the kernel in X of $(\sigma \otimes \sigma) \circ r$ where r is the Hopf-map defined in (3.26) and σ is the symbol map and let \mathcal{R} be the vector space generated by R and RF .

Note that R is a $*$ -ideal in X and

$$a_-, b_- b_+ (= q^2 b_+ b_-), b_- b_+^* \text{ are in } \mathcal{R}.$$

By construction and Theorem 3.3.4, any $T \in \mathcal{R}$ satisfies (3.48) and \mathcal{R} is invariant by F .

Moreover, by (3.15), $[b_-, b_-^*] \in R$, so by (3.12) and (3.18), $a_+^* a_+ - q^2 a_+ a_+^* - (1 - q^2) \in R$ and by (3.19), $q a_+ b_- - b_- a_+ \in R$.

It is interesting to quote, thanks to Theorem 3.3.4 that if $x \in R$, then $\oint F x |\mathcal{D}|^{-1} = 0$ while a priori, $\oint x |\mathcal{D}|^{-1} \neq 0$.

Note that $F \in \Psi^0(\mathcal{A})$ also satisfies (3.48) by Theorem 3.3.4 while $F \notin \mathcal{R}$ since $F^2 = 1$.

Moreover other elements are in \mathcal{R} like for instance $d(b^* b) = d(bb^*)$:

$$\begin{aligned} \delta(bb^*) &= -\delta(aa^*) = -\delta a a^* - a \delta a^* = -(a_+ - a_-)(a_+^* + a_-^*) - (a_+ + a_-)(a_-^* - a_+^*) \\ &= 2(a_+ a_-^* - a_- a_+^*) \end{aligned}$$

is in R since $a_- \in R$ yielding $d(bb^*) \in RF$.

We do not know if \mathcal{R} is equal to the subset of the algebra generated by \mathcal{B} and $\mathcal{B}F$ satisfying (3.48).

Lemma 3.5.7. \mathcal{R} is a $*$ -ideal in $\Psi^0(\mathcal{A})$ which is invariant by F, d, δ .

Proof. Since R is an ideal in $X = \mathcal{B} \bmod OP^{-\infty}$ (see Remark 3.3.3), \mathcal{R} appears to be an ideal in $\Psi^0(\mathcal{A}) \subset$ algebra generated by \mathcal{B} and $\mathcal{B}F$. Since \mathcal{R} is invariant by F , its invariance by d follows from its invariance by δ which is true on the generators of R . \square

Note that, according to Theorem 3.4.13, $\oint da |\mathcal{D}|^{-2} = \oint da |\mathcal{D}|^{-3} = 0$ while $\oint a^* da |\mathcal{D}|^{-3} = 2$ which emphasizes the importance the quantifiers "for all" in (3.48).

Lemma 3.5.8. *For any $t \in \Psi_0^0(\mathcal{A})$ and $T \in \mathcal{R}$, we have $\oint t T |\mathcal{D}|^{-1} = \oint T t |\mathcal{D}|^{-1}$.*

Proof. For any $t \in \mathcal{B}$, we have $\oint T t |\mathcal{D}|^{-1} = \oint t T |\mathcal{D}|^{-1} + \oint T |\mathcal{D}|^{-1} \delta(t) |\mathcal{D}|^{-1}$ and moreover $\oint T |\mathcal{D}|^{-1} \delta(t) |\mathcal{D}|^{-1} = \oint T \delta(t) |\mathcal{D}|^{-2} - \oint T \delta^2(t) |\mathcal{D}|^{-3}$ which comes from

$$\begin{aligned} |\mathcal{D}|^{-1} \delta(t) |\mathcal{D}|^{-1} &= \delta(t) |\mathcal{D}|^{-2} + [|\mathcal{D}|^{-1}, \delta(t)] |\mathcal{D}|^{-1} = \delta(t) |\mathcal{D}|^{-2} - |\mathcal{D}|^{-1} \delta^2(t) |\mathcal{D}|^{-2} \\ &= \delta(t) |\mathcal{D}|^{-2} - \delta^2(t) |\mathcal{D}|^{-3} + |\mathcal{D}|^{-1} \delta^3(t) |\mathcal{D}|^{-3}. \end{aligned}$$

So we get the result because T satisfies (3.48). \square

Lemma 3.5.9. *If \simeq means equality up to the ideal \mathcal{R} , the following rules with $d(\cdot) = [\mathcal{D}, \cdot]$ of the first-order differential calculus hold (suppressing $\underline{\pi}$)*

$$\begin{array}{llll} a da \simeq da a, & a^* da \simeq -da^* a, & b da \simeq q da b, & b^* da \simeq q da b^*, \\ a da^* \simeq -da a^*, & a^* da^* \simeq da^* a^*, & b da^* \simeq q^{-1} da^* b, & b^* da^* \simeq q^{-1} da^* b^*, \\ a db \simeq q^{-1} db a, & a^* db \simeq q db a^*, & b db \simeq db b, & b^* db \simeq db b^* \simeq -b db^*, \\ a db^* \simeq q^{-1} db^* a, & a^* db^* \simeq q db^* a^*, & b db^* \simeq db^* b \simeq -b^* db, & b^* db^* \simeq db^* b^*. \end{array}$$

Moreover

$$a^* da - q^2 da a^* \simeq (1 - q^2) F, \quad q^2 a da^* - da^* a \simeq (1 - q^2) F. \quad (3.49)$$

Proof. The table follows from relations (3.4) and Lemma 3.3.2 with (3.24) (one can also use (3.11)). For instance, since $a_- \in \mathcal{R}$, using the fact that \mathcal{R} is invariant by F ,

$$\begin{aligned} b da &= (b_+ + b_-)(a_+ - a_-) F \simeq (b_+ + b_-)(a_+ + a_-) F = ba F = q ab F \simeq q(a_+ - a_-) F b \\ &= q da b \end{aligned}$$

or similarly, $a^* da = (a_+^* + a_-^*)(a_+ - a_-) F \simeq (a_+^* - a_-^*)(a_+ + a_-) F = -da^* a$.

The second equivalence of (3.49) is just the adjoint of the first one that we prove now:

$$\begin{aligned} a^* da - q^2 da a^* &= (a_+^* + a_-^*)(a_+ - a_-) F - q^2 (a_+ - a_-) F (a_+^* + a_-^*) \\ &\simeq (a_+^* + a_-^*)(a_+ + a_-) F - q^2 (a_+ + a_-) (a_+^* + a_-^*) F = (a^* a - q^2 a a^*) F \\ &= (1 - q^2) F. \end{aligned} \quad \square$$

Remark 3.5.10. *The rule written above remains if dx is replaced by $\delta(x)$ and F by 1.*

Working modulo \mathcal{R} simplifies the writing of a one-form:

Lemma 3.5.11. *(i) Every one-form A can be, up to elements from \mathcal{R} , presented as*

$$A \simeq x_a da + da^* x_{a^*} + x_b db + db^* x_{b^*},$$

where all x_* are the elements of \mathcal{A} .

(ii) When A is selfadjoint, A can be written up to \mathcal{R} (not in a unique way, though) as

$$A \simeq x_a da - da^* (x_a)^* + x_b db - db^* (x_b)^*,$$

where x_a, x_b are arbitrary elements of \mathcal{A} .

Proof. (i) A basis for one-forms consists of the following forms: $a^\alpha b^\beta (b^*)^\gamma d(a^{\alpha'} b^{\beta'} (b^*)^{\gamma'})$, where $\alpha, \alpha' \in \mathbb{Z}$ and $\beta, \gamma, \beta', \gamma' \in \mathbb{N}$.

Using the Leibniz rule and the commutation rules within the algebra (up to the \mathcal{R} according to Lemma 3.5.9), we reduce the problem to the case of the forms: $(a^\alpha b^\beta (b^*)^\gamma) dx (a^{\alpha'} b^{\beta'} (b^*)^{\gamma'})$, where x can be either of the generators a, a^*, b, b^* . If $x = b$ or $x = b^*$, the straightforward application of the rules of the differential calculus leads to the answer that the one-form could be expressed as: $a^\alpha b^\beta (b^*)^\gamma db$ and $db^* a^\alpha b^\beta (b^*)^\gamma$.

Similar considerations for the case $x = a, a^*$ lead to the remaining terms.

Note that the presentation is not unique, since there still might remain terms, which are in \mathcal{R} , for example: $b^* db + db^* b = d(bb^*) \in \mathcal{R}$.

(ii) is direct. \square

Next we can start explicit calculation of the integrals, beginning with the tadpole terms.

Application of the Leibniz rule yields a presentation of one-forms which is different from the one of the previous lemma. Each δ -one-form could be expressed, as a finite sum of the terms $x\delta(z)y$, where z is one of the generators a, a^*, b, b^* and x, y are some elements of the algebra $\mathcal{A}(SU_q(2))$.

Proposition 3.5.12. *For all $x, y \in \mathcal{A}(SU_q(2))$ and $z \in \{a, a^*, b, b^*\}$ we have*

$$\oint x\delta(z)y |\mathcal{D}|^{-1} = \oint yx\delta(z) |\mathcal{D}|^{-1} + \oint x\delta(z)\delta(y) |\mathcal{D}|^{-2} - \oint x\delta(z)\delta^2(y) |\mathcal{D}|^{-3}.$$

Proof. This is just the application of the trace property of the noncommutative integral, together with the identity: $|\mathcal{D}|^{-1}\delta(z)|\mathcal{D}|^{-1} = -[|\mathcal{D}|^{-1}, z]$. \square

Remark 3.5.13. *The computation of tadpole-like integrals is reduced to the following integrals: $\oint x\delta(z)|\mathcal{D}|^{-1}$ for all generators z and the integrals of higher order in $|\mathcal{D}|^{-2}$. However, the calculations of higher-order terms simplify a lot when we use the relations which hold up to the ideal \mathcal{R} : this erases parts of the integral depending on $|\mathcal{D}|^{-2}$ and $|\mathcal{D}|^{-3}$. Thus, beside $\oint x\delta(z) |\mathcal{D}|^{-1}$, we only need to compute $\oint x\delta(z)\delta(z') |\mathcal{D}|^{-2}$ where z and z' are generators, since all the $|\mathcal{D}|^{-3}$ integrals have already been explicitly computed in section 4.6 (these integrals do not depend on q).*

Besides the tadpole, the only integrals that need to be computed are $\oint A |\mathcal{D}|^{-2}$ and $\oint A^2 |\mathcal{D}|^{-2}$ where A is a δ -1-form. Working modulo \mathcal{R} and using again Leibniz rule, we only need to compute $\oint x\delta(z) |\mathcal{D}|^{-2}$ and the previous integrals $\oint x\delta(z)\delta(z') |\mathcal{D}|^{-2}$.

Operators L_q and M_q

In the notation $v_{l,m}^j$ of \mathcal{H} , we have already used the j dependence in (3.8) with $J_q v_{m,l}^j := q^j v_{m,l}^j$.

Let L_q and M_q be the similar diagonal operators

$$\begin{aligned} L_q v_{m,l}^j &:= q^{2l} v_{m,l}^j, \\ M_q v_{m,l}^j &:= q^{2m} v_{m,l}^j. \end{aligned}$$

We immediately get

Lemma 3.5.14. For $n \in \mathbb{N}^*$, $\oint (L_q)^n |\mathcal{D}^{-2}| = \oint (M_q)^n |\mathcal{D}^{-2}| = \frac{2}{1-q^{2n}}$.

Proof. We have

$$\begin{aligned} \text{Tr} (L_q^n |\mathcal{D}|^{-2-s}) &= \sum_{2j=0}^{\infty} \sum_{m=0}^{2j} \sum_{l=0}^{2j+1} \langle v_{m,l}^j, L_q^n |\mathcal{D}|^{-2-s} v_{m,l}^j \rangle \\ &= \sum_{2j=0}^{\infty} (2j+1) \frac{1-q^{2n(2j+2)}}{1-q^{2n}} d_{j+}^{-2-s} + \sum_{2j=0}^{\infty} (2j+1) \frac{1-q^{2n(2j+2)}}{1-q^{2n}} d_j^{-2-s} \\ &\sim_0 \frac{1}{1-q^{2n}} \left(\zeta(s+1, \frac{3}{2}) + \zeta(s+1, \frac{1}{2}) \right) \sim_e \frac{2}{1-q^{2n}} \zeta(s+1). \end{aligned}$$

where \sim_0 means modulo a function holomorphic at 0. This gives the result for L_q^n and a similar computation can be done for M_q^n . \square

The interest of these operators stems from

Lemma 3.5.15. We have $L_q M_q \in \mathcal{R}$. Moreover,

$$\begin{aligned} b \delta b^* &\simeq M_q - L_q, \quad b^* \delta b \simeq L_q - M_q, \quad b b^* \simeq L_q + M_q, \\ a \delta(a^*) &\simeq -a a^* \simeq L_q + M_q - 1, \quad a^* \delta a \simeq a^* a \simeq 1 - q^2(L_q + M_q), \\ da da^* &\simeq L_q + M_q - 1, \quad da^* da \simeq q^2(L_q + M_q) - 1, \\ b^{n-2} (b^*)^n db db &\simeq (L_q)^n + (M_q)^n, \\ b^{n-1} (b^*)^{n-1} db db^* &\simeq -(L_q)^n - (M_q)^n, \\ b^n (b^*)^{n-2} db^* db^* &\simeq (L_q)^n + (M_q)^n. \end{aligned}$$

Proof. Since $L_q M_q = q^2 a_- a_-^* \in \mathcal{R}$, we compute up to the ideal \mathcal{R}

$$b \delta b^* = (b_+ + b_-)(b_-^* - b_+^*) \simeq -b_+ b_+^* + b_- b_-^* = M_q - L_q + L_q M_q (1 - q^2) \simeq M_q - L_q$$

and similarly for the other relations. \square

Automorphisms of the algebra and symmetries of integrals

Proposition 3.5.16. For any $n \in \mathbb{N}^*$,

$$\begin{aligned} \oint (b b^*)^n |\mathcal{D}|^{-1} &= \frac{-2(1+q^{2n})}{(1-q^{2n})^2}, \\ \oint (b b^*)^n b^* \delta b |\mathcal{D}|^{-1} &= \oint (b b^*)^n b \delta b^* |\mathcal{D}|^{-1} = \frac{2}{1-q^{2n+2}}, \\ \oint (b b^*)^n a \delta a^* |\mathcal{D}|^{-1} &= \frac{-2q^{4n+2} - 2q^{4n} - 2q^{2n+2} + 6q^{2n}}{(1-q^{2n})^2(1-q^{2n+2})}, \\ \oint (b b^*)^n a^* \delta a |\mathcal{D}|^{-1} &= \frac{6q^{2n+2} - 2q^{2n} - 2q^2 - 2}{(1-q^{2n})^2(1-q^{2n+2})}. \end{aligned}$$

Note that the knowledge of these integrals is enough for the computation of any term of the form $\oint x \delta(z) |\mathcal{D}|^{-1}$, where z is a generator, since any other δ -one-form will be unbalanced.

To show this proposition, we will use few symmetries, properties of the ideal \mathcal{R} and replacement of δ -one-forms in terms of L_q, M_q as above.

Let U be the following unitary operator on the Hilbert space:

$$U v_{m,l}^{j\uparrow} = (-1)^{m+l} v_{l,m}^{j+\downarrow}, \quad U v_{m,l}^{j\downarrow} = (-1)^{m+l} v_{l,m}^{j-\uparrow}.$$

Then, by explicit computations we have

$$U^* a U = a, \quad U^* a^* U = a^*, \quad U^* b U = b^*, \quad U^* b^* U = b, \quad \text{and} \quad U^* \mathcal{D} U = -\mathcal{D}.$$

Lemma 3.5.17. *Each noncommutative integral (3.41) of an element of the algebra or differential forms is (up to sign) invariant under the algebra automorphism ρ defined by*

$$\rho(a) := a, \quad \rho(a^*) := a^*, \quad \rho(b) := b^*, \quad \rho(b^*) := b. \quad (3.50)$$

Proof. For any homogeneous polynomial p and any $k \in \mathbb{N}$,

$$\begin{aligned} \oint p(a, a^*, b, b^*, \mathcal{D}) \mathcal{D}^{-k} &= \oint U^* p(a, a^*, b, b^*, \mathcal{D}) \mathcal{D}^{-k} U \\ &= (-1)^k \oint p(U^* a U, U^* a^* U, U^* b U, U^* b^* U, U^* \mathcal{D} U) \mathcal{D}^{-k} \\ &= (-1)^{k+d} \oint p(\rho(a), \rho(a^*), \rho(b), \rho(b^*), \mathcal{D}) \mathcal{D}^{-k}, \end{aligned}$$

where d is the degree of p with respect to \mathcal{D} . □

Corollary 3.5.18. *For any $n \in \mathbb{N}$, $\oint (bb^*)^n b^* db \mathcal{D}^{-1} = \oint (bb^*)^n b db^* \mathcal{D}^{-1}$.*

Lemma 3.5.19. *For any $x, y \in \Psi^0(\mathcal{A})$,*

$$\begin{aligned} (i) \quad & \oint xy |\mathcal{D}|^{-1} = \oint yx |\mathcal{D}|^{-1} + \oint x\delta(y) |\mathcal{D}|^{-2} - \oint x\delta^2(y) |\mathcal{D}|^{-2}. \\ (ii) \quad & \oint z x \mathcal{D}^{-1} y \mathcal{D}^{-1} = \oint z xy \mathcal{D}^{-2}, \text{ if } z \in \mathcal{A} \text{ contains } b \text{ or } b^*. \end{aligned}$$

Proof. (i) is direct consequence of the trace property of \oint and the fact that OP^{-4} operators are trace-class.

(ii) We calculate:

$$\begin{aligned} \oint z x \mathcal{D}^{-1} y \mathcal{D}^{-1} &= \oint z x (y \mathcal{D}^{-1} - \mathcal{D}^{-1} [\mathcal{D}, y] \mathcal{D}^{-1}) \mathcal{D}^{-1} = \oint z xy \mathcal{D}^{-2} - \oint z x \mathcal{D}^{-1} [\mathcal{D}, y] \mathcal{D}^{-2} \\ &= \oint z xy \mathcal{D}^{-2}. \end{aligned}$$

The last step is based on the observation that any integral with \mathcal{D}^{-3} vanishes if the expression integrated contains b or b^* . □

Lemma 3.5.20. *For any $n \in \mathbb{N}$,*

- (i) $\oint (bb^*)^n b^* db \mathcal{D}^{-1} = \frac{2}{1-q^{2n+2}}.$
- (ii) $\oint (bb^*)^n d(bb^*) \mathcal{D}^{-1} = 0.$
- (iii) $\oint (bb^*)^n |\mathcal{D}|^{-1} = \frac{-2(1+q^{2n})}{(1-q^{2n})^2}.$

Proof. (i) With $n > 1$, we begin with $\oint d((bb^*)^n) \mathcal{D}^{-1} = 0$, which follows directly from the trace property of the noncommutative integral. Expanding the expression using the Leibniz rule and the commutation

$$x\mathcal{D}^{-1} = \mathcal{D}^{-1}x + \mathcal{D}^{-1}[\mathcal{D}, x]\mathcal{D}^{-1}, \quad (3.51)$$

we obtain

$$\begin{aligned} 0 &= \sum_{k=0}^{n-1} \oint b^k db b^{n-k-1} (b^*)^n \mathcal{D}^{-1} + \sum_{k=0}^{n-1} \oint b^n (b^*)^k db^* (b^*)^{n-k-1} \mathcal{D}^{-1} \\ &= n \left(\oint b^{n-1} (b^*)^n db \mathcal{D}^{-1} + \oint b^n (b^*)^{n-1} db^* \mathcal{D}^{-1} \right) \\ &\quad + \sum_{k=0}^{n-1} \oint \left(b^k db \mathcal{D}^{-1} d(b^{n-k-1} (b^*)^n) \mathcal{D}^{-1} + b^n (b^*)^k db^* \mathcal{D}^{-1} d((b^*)^{n-k-1}) \mathcal{D}^{-1} \right). \end{aligned}$$

Using Lemma 3.5.19,

$$\begin{aligned} 0 &= n \oint (bb^*)^{n-1} (b^* db + b db^*) \mathcal{D}^{-1} \\ &\quad + \oint \left(\frac{1}{2} n(n-1) b^{n-2} (b^*)^n db db + n^2 b^{n-1} (b^*)^{n-1} db db^* + \frac{1}{2} n(n-1) b^n (b^*)^{n-2} db^* db^* \right) \mathcal{D}^{-2}. \end{aligned}$$

The integrals with \mathcal{D}^{-2} could be easily calculated when we restrict ourselves to calculations modulo the ideal \mathcal{R} :

$$n \oint (bb^*)^{n-1} (b^* db + b db^*) \mathcal{D}^{-1} = -2(n(n-1) - 2n^2 + n(n-1)) \frac{1}{1-q^{2n}} = 4n \frac{1}{1-q^{2n}}.$$

Hence $\oint (bb^*)^{n-1} (b^* db + b db^*) \mathcal{D}^{-1} = \frac{4}{1-q^{2n}}$, which together with Corollary 3.5.18 proves (i).

(ii) In a similar way, $\oint (bb^*)^{n-1} d(bb^*) \mathcal{D}^{-1} = 0 = \oint (bb^*)^{n-1} d(aa^*) \mathcal{D}^{-1}$ implies:

$$\begin{aligned} 0 &= \sum_{k=0}^{n-1} (bb^*)^{n-k-1} d(bb^*) (bb^*)^k \mathcal{D}^{-1} \\ &= n \oint (bb^*)^{n-1} d(bb^*) \mathcal{D}^{-1} + \frac{1}{2} n(n-1) \oint (bb^*)^{n-2} d(bb^*) d(bb^*) \mathcal{D}^{-2} \\ &= n \oint (bb^*)^{n-1} d(bb^*) \mathcal{D}^{-1}, \end{aligned}$$

where in the last step we used that $d(bb^*) \in \mathcal{R}$. The identity (ii) now follows from the equality $aa^* = 1 - bb^*$.

(iii) Using Lemma 3.5.19, we get

$$A_n := \oint (bb^*)^n |\mathcal{D}|^{-1} = \oint (bb^*)^n (aa^* + bb^*) |\mathcal{D}|^{-1}$$

and we push now a^* through $|\mathcal{D}|^{-1}$ and from cyclicity of the trace through $(bb^*)^n$,

$$= A_{n+1} + \oint (bb^*)^n q^{2n} a^* a |\mathcal{D}|^{-1} + \oint (bb^*)^n q^{2n} a \delta(a^*) |\mathcal{D}|^{-2}$$

the last term being calculated explicitly, since up to ideal \mathcal{R} , $a\delta(a^*) \simeq L_q + M_q - 1$,

$$= A_{n+1}(1 - q^{2n+2}) + q^{2n} A_n + 4 \left(\frac{1}{1 - q^{2n+2}} - \frac{1}{1 - q^{2n}} \right),$$

which leads to

$$A_n(1 - q^{2n}) + \frac{4}{1 - q^{2n}} = A_{n+1}(1 - q^{2n+2}) + \frac{4}{1 - q^{2n+2}}.$$

Assuming $A_n = \frac{f_n}{(1 - q^{2n})^2}$ we have $\frac{f_{n+4}}{1 - q^{2n}} = \frac{f_{n+1+4}}{1 - q^{2n+2}}$, and taking into account that $A_0 = -2 \frac{1+q^2}{(1-q^2)^2}$, we obtain $A_n = -2 \frac{1+q^{2n}}{(1-q^{2n})^2}$. \square

Finally, to get Proposition 3.5.16, it remains to prove

Lemma 3.5.21. *For $n \geq 1$,*

$$\begin{aligned} \oint (bb^*)^n a da^* \mathcal{D}^{-1} &= \frac{-2q^{4n+2} - 2q^{4n} - 2q^{2n+2} + 6q^{2n}}{(1 - q^{2n})^2(1 - q^{2n+2})}, \\ \oint (bb^*)^n a^* da \mathcal{D}^{-1} &= \frac{6q^{2n+2} - 2q^{2n} - 2q^2 - 2}{(1 - q^{2n})^2(1 - q^{2n+2})}. \end{aligned}$$

Proof. First, using the Leibniz rule, (3.51) and Lemma 3.5.19 we have (for $n \geq 1$)

$$\oint (bb^*)^n a da^* \mathcal{D}^{-1} = -q^{2n} \oint (bb^*)^n a^* da - \oint (bb^*)^n da da^* \mathcal{D}^{-2}.$$

Further, we use the identity (3.42):

$$\oint (bb^*)^n (a^* da + q^2 a da^* + q^2 b db^* + q^2 b^* db) \mathcal{D}^{-1} = (1 - q^2) \oint (bb^*)^n |\mathcal{D}|^{-1}.$$

taking into account that $F\mathcal{D} = |\mathcal{D}|$.

These equations give together a system of linear equations

$$\begin{aligned} \oint (bb^*)^n a da^* \mathcal{D}^{-1} + q^{2n} \oint (bb^*)^n a^* da \mathcal{D}^{-1} &= -4 \left(\frac{1}{1 - q^{2n+2}} - \frac{1}{1 - q^{2n}} \right), \\ q^2 \oint (bb^*)^n a da^* \mathcal{D}^{-1} + \oint (bb^*)^n a^* da \mathcal{D}^{-1} &= -2(1 - q^2) \frac{1 + q^{2n}}{(1 - q^{2n})^2} - \frac{4q^2}{1 - q^{2n+2}} \end{aligned}$$

which is solved by the expressions stated in the lemma. \square

The noncommutative integrals at $|\mathcal{D}|^{-2}$

We need to separate this task into two problems. First, we shall calculate all integrals of the type $\oint x \delta(z) |\mathcal{D}|^{-2}$, with $x \in \mathcal{A}(SU_q(2))$ and z being one of the generators. The second problem is to calculate $\oint x \delta(y) \delta(z) |\mathcal{D}|^{-2}$, with both y and z being the generators $\{a, a^*, b, b^*\}$.

Lemma 3.5.22. *The only a priori non-vanishing integrals of the type $\oint x \delta(z) |\mathcal{D}|^{-2}$ are for $n \in \mathbb{N}$:*

$$\begin{aligned} \oint (bb^*)^n b^* \delta(b) |\mathcal{D}|^{-2} &= \oint (bb^*)^n b \delta(b^*) |\mathcal{D}|^{-2} = 0, \\ \oint (bb^*)^n a \delta(a^*) |\mathcal{D}|^{-2} &= \frac{4q^{2n}(1-q^2)}{(q^{2n+2}-1)(1-q^{2n})}, \quad n > 0 \\ \oint (bb^*)^n a^* \delta(a) |\mathcal{D}|^{-2} &= \frac{4(1-q^2)}{(1-q^{2n+2})(1-q^{2n})}. \end{aligned}$$

Proof. Since $a\delta(a^*) \simeq L_q + M_q - 1$ and $(bb^*)^n \simeq L_q^n + M_q^n$, we get

$$(bb^*)^n a\delta(a^*) \simeq L_q^{n+1} + M_q^{n+1} - L_q^n - M_q^n$$

and the second result is obtained from Lemma 3.5.14. The other integrals are computed in a similar way. \square

Lemma 3.5.23. *The only a priori non-vanishing integrals of the type $\oint x dy dz |\mathcal{D}|^{-2}$ are for $n \in \mathbb{N}$:*

$$\begin{aligned} \oint (bb^*)^n (b^*)^2 db db |\mathcal{D}|^{-2} &= \frac{4}{1-q^{2n+4}}, \\ \oint (bb^*)^n db db^* |\mathcal{D}|^{-2} &= \frac{4}{1-q^{2n+2}}, \\ \oint (bb^*)^n (a^* b^*)(da db) |\mathcal{D}|^{-2} &= 0, \\ \oint (bb^*)^n (ab^*)(da^* db) |\mathcal{D}|^{-2} &= 0, \\ \oint (bb^*)^n (a^* b)(da db^*) |\mathcal{D}|^{-2} &= 0, \\ \oint (bb^*)^n (ab)(da^* db^*) |\mathcal{D}|^{-2} &= 0, \\ \oint (bb^*)^n (da da^*) |\mathcal{D}|^{-2} &= \frac{4(q^{2n+2}-q^{2n})}{(1-q^{2n+2})(1-q^{2n})}, \quad n > 0 \\ \oint (bb^*)^n (da^* da) |\mathcal{D}|^{-2} &= \frac{4(q^2-1)}{(1-q^{2n+2})(1-q^{2n})}. \end{aligned}$$

Proof. This follows from Lemma 3.5.14 with the equivalences up to \mathcal{R} gathered in Lemma 3.5.15. \square

3.6 Examples of spectral action

It is clear from Theorem 3.4.3 that any one-form of the form ada , bdb , adb , a^*db , etc... do not contribute to the spectral action. Indeed, only the balanced parts of one-forms give a possibly nonzero term in the coefficients. We give in the following table the values of the terms $\oint A^n |\mathcal{D}|^{-p}$ and the full $\zeta_{\mathcal{D}_{\mathbb{A}}}(0)$ for a few examples.

TAB. 3.1 – Values of noncommutative integrals

\mathbb{A}	$\oint A \mathcal{D} ^{-3}$	$\oint A^2 \mathcal{D} ^{-3}$	$\oint A^3 \mathcal{D} ^{-3}$	$\oint A \mathcal{D} ^{-2}$	$\oint A^2 \mathcal{D} ^{-2}$	$\oint A \mathcal{D} ^{-1}$	$\zeta_{\mathcal{D}_{\mathbb{A}}}(0)$
a^*da	2	2	2	$\frac{4q^2}{q^2-1}$	$\frac{4q^2(q^2+2)}{q^4-1}$	$\frac{3q^2+1}{2(q^2-1)}$	$\frac{11q^4+36q^2+13}{3(q^4-1)}$
b^*db	0	0	0	0	$\frac{-4}{q^4-1}$	$\frac{-2}{q^2-1}$	$\frac{4q^2}{q^4-1}$
ada^*	-2	2	-2	$\frac{-4}{q^2-1}$	$\frac{4(2q^2+1)}{q^4-1}$	$\frac{q^2+3}{2(q^2-1)}$	$\frac{13q^4+36q^2+11}{3(q^4-1)}$
bdb^*	0	0	0	0	$\frac{-4}{q^4-1}$	$\frac{-2}{q^2-1}$	$\frac{4q^2}{q^4-1}$

1) Clearly the spectral action depends on q : for instance,

$$\mathcal{S}(\mathcal{D}_{a^*da}, \Phi, \Lambda) = 2\Phi_3 \Lambda^3 - 8\Phi_2 \Lambda^2 + \frac{q^2+15}{2(1-q^2)} \Phi_1 \Lambda^1 + \frac{11q^4+36q^2+13}{3(q^4-1)} \Phi(0).$$

2) Moreover, for $B := a\delta a^*$ and $A := B + B^*$, we get since $B \simeq B^* \pmod{\mathcal{R}}$,

$$\oint A^p |\mathcal{D}|^{-k} = 2^p \oint B^p |\mathcal{D}|^{-k}, \quad 1 \leq p \leq k \leq 3. \quad (3.52)$$

Thus the spectral action of the selfadjoint one-form $\mathbb{A} := ada^* + (ada^*)^*$ is

$$\mathcal{S}(\mathcal{D}_{\mathbb{A}}, \Phi, \Lambda) = 2\Phi_3 \Lambda^3 + 16\Phi_2 \Lambda^2 + \frac{q^2-33}{2(1-q^2)} \Phi_1 \Lambda^1 + \frac{122q^4+168q^2-2}{3(q^4-1)} \Phi(0).$$

3) When $B_n := (bb^*)^n b\delta b^*$, then by Lemma (3.5.15), $B_n \simeq B_n^*$, so for $A_n := B_n + B_n^*$, the equation (3.52) is still valid and $\oint B_n^p |\mathcal{D}|^{-k}$ are all zero but $\oint B_n |\mathcal{D}|^{-1} = \frac{2}{1-q^{2n+2}}$ and $\oint B_n^2 |\mathcal{D}|^{-2} = \frac{4}{1-q^{4n+4}}$, so

$$\mathcal{S}(\mathcal{D}_{\mathbb{A}_n}, \Phi, \Lambda) = 2\Phi_3 \Lambda^3 - \frac{1}{2} \Phi_1 \Lambda^1 + \frac{8}{1+q^{2n+2}} \Phi(0). \quad (3.53)$$

Remark that this spectral action still exists as $q \rightarrow 1$!

Note however that the symmetrization process (3.52) does not work in general, for instance if $B := a\delta b$ and $A := B + B^*$, then $\oint A^2 |\mathcal{D}|^{-1} = \frac{8(q^4-q^2-1)}{(1-q^4)^2}$ while $\oint B^2 |\mathcal{D}|^{-1} = 0$ or $\oint [B, B^*] |\mathcal{D}|^{-1} = \frac{4}{1-q^4}$.

4) The spectral action can be also independent of q : for instance, if $\mathbb{A} = \frac{1}{1-q^2} \xi(\mathcal{D})$ is the q -dependent selfadjoint one-form given in (3.44), then,

$$\mathcal{S}(\mathcal{D}_{\mathbb{A}}, \Phi, \Lambda) = 2\Phi_3 \Lambda^3 - 8\Phi_2 \Lambda^2 + \frac{15}{2} \Phi_1 \Lambda^1 - \frac{13}{3}.$$

3.7 The commutative sphere \mathbb{S}^3

Since $SU(2) \simeq \mathbb{S}^3$, we get a concrete spinorial representation of the algebra $\mathcal{A} := C^\infty(\mathbb{S}^3)$ on the same Hilbert space \mathcal{H} and same Dirac operator \mathcal{D} with (3.6) where $q = 1$ which means that q -numbers are trivial: $[\alpha] = \alpha$. So

$$\begin{aligned}\pi(a) |j\mu n\rangle &:= \alpha_{j\mu n}^+ |j^+ \mu^+ n^+\rangle + \alpha_{j\mu n}^- |j^- \mu^+ n^+\rangle, \\ \pi(b) |j\mu n\rangle &:= \beta_{j\mu n}^+ |j^+ \mu^+ n^-\rangle + \beta_{j\mu n}^- |j^- \mu^+ n^-\rangle, \\ \pi(a^*) |j\mu n\rangle &:= \tilde{\alpha}_{j\mu n}^+ |j^+ \mu^- n^-\rangle + \tilde{\alpha}_{j\mu n}^- |j^- \mu^- n^-\rangle, \\ \pi(b^*) |j\mu n\rangle &:= \tilde{\beta}_{j\mu n}^+ |j^+ \mu^- n^+\rangle + \tilde{\beta}_{j\mu n}^- |j^- \mu^- n^+\rangle\end{aligned}\tag{3.54}$$

where

$$\begin{aligned}\alpha_{j\mu n}^+ &:= \sqrt{j + \mu + 1} \begin{pmatrix} \frac{\sqrt{j+n+3/2}}{2j+2} & 0 \\ \frac{\sqrt{j-n+1/2}}{(2j+1)(2j+2)} & \frac{\sqrt{j+n+1/2}}{2j+1} \end{pmatrix}, \\ \alpha_{j\mu n}^- &:= \sqrt{j - \mu} \begin{pmatrix} \frac{\sqrt{j-n+1/2}}{2j+1} & -\frac{\sqrt{j+n+1/2}}{2j(2j+1)} \\ 0 & \frac{\sqrt{j-n-1/2}}{2j} \end{pmatrix}, \\ \beta_{j\mu n}^+ &:= \sqrt{j + \mu + 1} \begin{pmatrix} \frac{\sqrt{j-n+3/2}}{2j+2} & 0 \\ -\frac{\sqrt{j+n+1/2}}{(2j+1)(2j+2)} & \frac{\sqrt{j-n+1/2}}{2j+1} \end{pmatrix}, \\ \beta_{j\mu n}^- &:= \sqrt{j - \mu} \begin{pmatrix} -\frac{\sqrt{j+n+1/2}}{2j+1} & -\frac{\sqrt{j-n+1/2}}{2j(2j+1)} \\ 0 & -\frac{\sqrt{j+n-1/2}}{2j} \end{pmatrix}\end{aligned}$$

with $\tilde{\alpha}_{j\mu n}^\pm := (\alpha_{j^\mp \mu^\mp n^\mp}^\mp)^*$, $\tilde{\beta}_{j\mu n}^\pm := (\beta_{j^\mp \mu^\mp n^\mp}^\mp)^*$.

Note that the representation on the vectors $v_{m,l}^j$ is not as convenient as in (3.7).

One can check that the generators $\pi(a)$, $\pi(b)$ and their adjoint commute and that $[x, [\mathcal{D}, y]] = 0$ for any $x, y \in \mathcal{A}$.

3.7.1 Translation of Dirac operator

In general the Dirac operator is defined in a more symmetric way than what we did. So, although not absolutely necessary here, we define for the interested reader the unbounded self-adjoint translated operator \mathcal{D}' on \mathcal{H} by the constant λ as

$$\mathcal{D}' := \mathcal{D} + \lambda.$$

For instance, this gives for $\lambda = -\frac{1}{2}$ in the case of \mathbb{S}^3 , see [80], $\mathcal{D}' v_{m,l}^j = (2j+1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v_{m,l}^j$ so $v_{m,l}^j$ is an eigenvector of $|\mathcal{D}'|$. We define $D := \mathcal{D} + P_0$ and $D' := \mathcal{D}' + P'_0$ where P_0 is the projection on $\text{Ker } \mathcal{D}$ and P'_0 is the projection on $\text{Ker } \mathcal{D}'$.

As the following lemma shows, the computation of the noncommutative integrals involving \mathcal{D} can be reduced to the computation of certain noncommutative integrals involving \mathcal{D}' :

Lemma 3.7.1. *If $\oint' T := \text{Res}_{s=0} \text{Tr} (T|D'|^{-s})$, then for any 1-form \mathbb{A} on a spectral triple of dimension n ,*

$$\begin{aligned} \oint A|D|^{-(n-2)} &= \oint' A|D'|^{-(n-2)} + \lambda(n-2) \oint' A\mathcal{D}'|D'|^{-n} + \lambda^2 \frac{(n-1)(n-2)}{2} \oint' A|D'|^{-n}, \\ \oint A D^{-(n-2)} &= \oint' A D'^{-(n-2)} + \lambda(n-2) \oint' A D'^{-(n-1)} + \lambda^2 \frac{(n-1)(n-2)}{2} \oint' A D'^{-n}. \end{aligned}$$

Proof. Recall from Proposition 1.3.8 that for any pseudodifferential operator P ,

$$\oint P|D|^{-r} = \text{Res}_{s=0} \text{Tr} (P|D|^{-r}|D'|^{-s}).$$

Moreover by Lemma 1.3.3, for any $s \in \mathbb{C}$ and $N \in \mathbb{N}^*$

$$|D|^{-s} = |D'|^{-s} + \sum_{p=1}^N K_{p,s} Y^p |D'|^{-s} \mod OP^{-N-1-\Re(s)} \quad (3.55)$$

where $Y = \sum_{k=1}^N \frac{(-1)^{k+1}}{k} (-2\lambda D' + \lambda^2)^k D'^{-2k} \mod OP^{-N-1}$ and $K_{p,s}$ are complex numbers that can be explicitly computed. Precisely, we find $K_{p,s} = (-\frac{s}{2})^p V(p)$ where $V(p)$ is the volume of the p -simplex. Since the spectral dimension is n , we work modulo $OP^{-(n+1)}$, and for $s = n-2$, we get from (3.55): $|D|^{-(n-2)} = |D'|^{-(n-2)} + \lambda(n-2) \mathcal{D}'|D'|^{-n} + \lambda^2 \frac{(n-1)(n-2)}{2} |D'|^{-n} \mod OP^{-(n+1)}$.

As a consequence, we have for $P \in OP^0$ (the OP^0 spaces are the same for \mathcal{D} or \mathcal{D}'),

$$\oint P|D|^{-(n-2)} = \oint' P|D'|^{-(n-2)} + \lambda(n-2) \oint' P\mathcal{D}'|D'|^{-n} + \lambda^2 \frac{(n-1)(n-2)}{2} \oint' P|D'|^{-n}.$$

Since A and AF are in OP^0 , we get both formulae. \square

3.7.2 Tadpole and spectral action on \mathbb{S}^3

We consider now the commutative spectral triple $(C^\infty(\mathbb{S}^3), \mathcal{H}, \mathcal{D})$. It is 1-summable since $\langle\langle j\mu n s | [F, \pi(x)] | j\mu n s \rangle\rangle = 0$ when $x = a, a^*, b, b^*$ for any $j, \mu, n, s = \uparrow, \downarrow$.

All integrals of above lemma are zero for \mathbb{S}^3 :

Proposition 3.7.2. *There is no tadpole of any order on the commutative real spectral triple $(C^\infty(\mathbb{S}^3), \mathcal{H}, \mathcal{D})$. More generally, for any one-form \mathbb{A} , $\oint \mathbb{A} F |\mathcal{D}|^{-p} = 0$ for $p \in \mathbb{N}$.*

Proof. Since the representation is real, that is any matrix elements of the generators are real, so must be the trace of $\mathbb{A} F |\mathcal{D}|^{-p}$. Hence $\oint \mathbb{A} F |\mathcal{D}|^{-p} = \oint \mathbb{A}^* F |\mathcal{D}|^{-p}$.

The reality operator J introduced in (3.25) satisfies, when $q = 1$, the commutative relation $JxJ^{-1} = x^*$ for $x \in \mathcal{A}$. Thus $J\mathbb{A}J^{-1} = -\mathbb{A}^*$ and $\oint \mathbb{A} F |\mathcal{D}|^{-p} = \oint J(\mathbb{A}^* F |\mathcal{D}|^{-p}) J^{-1} = -\oint \mathbb{A} F |\mathcal{D}|^{-p}$ and $\oint \mathbb{A} F |\mathcal{D}|^{-p} = 0$. \square

For any selfadjoint one-form \mathbb{A} , $\mathcal{D}_{\mathbb{A}} := \mathcal{D} + \tilde{\mathbb{A}} = \mathcal{D}$. Thus, the spectral action for the real spectral triple $(C^\infty(\mathbb{S}^3), \mathcal{H}, \mathcal{D})$ for $\mathcal{D}_{\mathbb{A}}$ is trivialized by

$$\mathcal{S}(\mathcal{D}_{\mathbb{A}}, \Phi, \Lambda) = 2\Phi_3 \Lambda^3 - \frac{1}{2} \Phi_1 \Lambda^1 + \mathcal{O}(\Lambda^{-1}). \quad (3.56)$$

But it is more natural to compare with the spectral action of $\mathcal{D} + \mathbb{A}$. This is obtained respectively from Lemma 3.4.18 and general heat kernel approach [65]:

$$\mathcal{S}(\mathcal{D} + \mathbb{A}, \Phi, \Lambda) = 2 \Phi_3 \Lambda^3 + \oint |\mathcal{D} + \mathbb{A}|^{-1} \Phi_1 \Lambda^1 + \mathcal{O}(\Lambda^{-1})$$

since all terms of (1.6) in Λ^{n-k} are zero for k odd and $\zeta_{\mathcal{D}+\mathbb{A}}(0) = 0$ when n is odd: as a verification, $\oint |\mathcal{D} + \mathbb{A}|^{-2}$ is zero according to Lemma 1.3.10, Lemmas 3.4.18 and Proposition 3.7.2. Similarly, $\zeta_{\mathcal{D}+\mathbb{A}}(0) = 0$ because in (1.11), all terms with k odd are zero (same proof as in Proposition 3.7.2) but for k even, it is not that easy to show that $\oint \mathbb{A} \mathcal{D}^{-1} \mathbb{A} \mathcal{D}^{-1} = 0$.

Moreover, the curvature term does not depend on \mathbb{A} :

Lemma 3.7.3. *For any one-form \mathbb{A} on a commutative spectral triple of dimension n based on a compact Riemannian spin^c manifold without boundary, we have*

$$\oint |\mathcal{D} + \mathbb{A}|^{-(n-2)} = \oint |\mathcal{D}|^{-(n-2)}. \quad (3.57)$$

Proof. Follows from [68, first formula page 511] with $\rho := \mathbb{A} = \mathbb{A}^*$, $N(\rho) = \rho$ (the constraint $J\rho J^{-1} = \pm\rho$ is not used).

One can also use [37, Proposition 1.149]. \square

From Lemma 1.3.10, $\oint |\mathcal{D} + \mathbb{A}|^{-(n-2)} = \oint |\mathcal{D}|^{-(n-2)} + \frac{n(n-2)}{4} \oint (\mathbb{A}F)^2 |\mathcal{D}|^{-3} + \frac{(n-2)^2}{4} \oint \mathbb{A}^2 |\mathcal{D}|^{-3}$ using $X := \mathbb{A}\mathcal{D} + \mathcal{D}\mathbb{A} + \mathbb{A}^2$ and $[|\mathcal{D}|, \mathbb{A}] \in OP^0$, but again, it is not that easy to show that the last two terms cancel: for instance here, for $\mathbb{B} = b[\mathcal{D}, b^*]$, we obtain by direct computation (using the easiest translated Dirac operator \mathcal{D}')

$$\begin{aligned} \text{Tr}(\mathbb{B}^2 |\mathcal{D}'|^{-3-s}) &= \text{Tr}((\mathbb{B}^*)^2 |\mathcal{D}'|^{-3-s}) = \frac{1}{2} \text{Tr}(\mathbb{B}\mathbb{B}^* |\mathcal{D}'|^{-3-s}) = \frac{1}{2} \text{Tr}(\mathbb{B}^* \mathbb{B} |\mathcal{D}'|^{-3-s}) \\ &= \frac{4}{3} \sum_{2j \in \mathbb{N}} \frac{j+1}{(2j+1)^{2+s}}, \end{aligned}$$

so $\oint \mathbb{B}^2 |\mathcal{D}'|^{-3} = \frac{2}{3}$. Similarly, one checks that $\oint (\mathbb{B}F)^2 |\mathcal{D}|^{-3} = \frac{1}{2} \oint \mathbb{B}F\mathbb{B}^*F |\mathcal{D}|^{-3} = -\frac{2}{9}$. Thus if $\mathbb{A} := \mathbb{B} + \mathbb{B}^*$, $\oint \mathbb{A}^2 |\mathcal{D}|^{-3} = \oint \mathbb{A}^2 |\mathcal{D}'|^{-3} = 4$ and $\oint (\mathbb{A}F)^2 |\mathcal{D}|^{-3} = -\frac{4}{3}$ which yields (3.57).

Thus for any one-form \mathbb{A} on the 3-sphere,

$$\mathcal{S}(\mathcal{D} + \mathbb{A}, \Phi, \Lambda) = 2 \Phi_3 \Lambda^3 - \frac{1}{2} \Phi_1 \Lambda^1 + \mathcal{O}(\Lambda^{-1}, \mathbb{A})$$

which as (3.56) is not identical to (3.53) which contains a nonzero constant term Λ^0 for $q = 1$.

3.8 Appendix

A. Proof of Lemma 3.4.13

(i) Using same notations of Lemma 3.4.12, we obtain by definition of τ_1 ,

$$\tau_1(\pi_+(t_{k,p})) = \delta_{k,0} \delta_{p,0} \delta_{\alpha_1+\alpha_2-\alpha_3+\beta_1+\beta_2-\beta_3,0}, \quad (3.58)$$

$$\tau_1(\pi_-(u_{k,p})) = \delta_{\tilde{k},0} \delta_{\tilde{p},0} \delta_{\alpha_1-\alpha_2+\alpha_3+\beta_1-\beta_2+\beta_3,0}. \quad (3.59)$$

We get $\tau_1(\pi_+(t_{k,p})) \tau_1(\pi_-(u_{k,p})) = \delta_{k,0} \delta_{p,0} \delta_{\alpha_2,0} \delta_{\alpha_3,0} \delta_{\beta_2,0} \delta_{\beta_3,0} \delta_{\alpha_1,-\beta_1}$, so Lemma 3.4.12 gives the result.

(ii) Since $\pi_+(t_{k,p})\varepsilon_n = q_{k,p,n}^+ \varepsilon_{n+r_{k,p}^+}$ and $\pi_-(u_{k,p})\varepsilon_n = q_{k,p,n}^- \varepsilon_{n+r_{k,p}^-}$, we get,

$$\tau_0(\pi_+(t_{k,p})) = \delta_{r_{k,p}^+,0} \sum_{n=0}^{\infty} (q_{k,p,n}^+ - \delta_{k,0} \delta_{p,0} \delta_{\alpha_1+\alpha_2-\alpha_3+\beta_1+\beta_2-\beta_3,0}), \quad (3.60)$$

$$\tau_0(\pi_-(u_{k,p})) = \delta_{r_{k,p}^-,0} \sum_{n=0}^{\infty} (q_{k,p,n}^- - \delta_{k,0} \delta_{p,0} \delta_{\alpha_1-\alpha_2+\alpha_3+\beta_1-\beta_2+\beta_3,0}). \quad (3.61)$$

With (3.58) and (3.61) we get

$$\begin{aligned} \tau_1(\pi_+(t_{k,p})) \tau_0(\pi_-(u_{k,p})) &= \delta_{k,0} \delta_{p,0} \delta_{\alpha_2+\beta_2,\alpha_3+\beta_3} \delta_{\alpha_1,-\beta_1} \sum_{n=0}^{\infty} (\delta_{k,0} \delta_{p,0} q_{k,p,n}^- - \delta_{\alpha_3+\beta_3,0}) \\ &= \delta_{k,0} \delta_{p,0} \delta_{\alpha_2+\beta_2,\alpha_3+\beta_3} \delta_{\alpha_1,-\beta_1} w_1(\beta_1, \alpha_3 + \beta_3). \end{aligned}$$

Using (3.59) and (3.60),

$$\begin{aligned} \tau_0(\pi_+(t_{k,p})) \tau_1(\pi_-(u_{k,p})) &= \delta_{k,0} \delta_{p,0} \delta_{\alpha_2+\beta_2,\alpha_3+\beta_3} \delta_{\alpha_1,-\beta_1} \sum_{n=0}^{\infty} (\delta_{k,0} \delta_{p,0} q_{k,p,n}^+ - \delta_{\alpha_3+\beta_3,0}) \\ &= \delta_{k,0} \delta_{p,0} \delta_{\alpha_2+\beta_2,\alpha_3+\beta_3} \delta_{\alpha_1,-\beta_1} w_1(\beta_1, \alpha_3 + \beta_3). \end{aligned}$$

Lemma 3.4.12 yields the result.

B. Proof of Lemma 3.4.14

We have

$$\tau_1(\pi_+(t_{K,P})) = \delta_{K,0} \delta_{P,0} \delta_{A_1+A_2-A_3+B_1+B_2-B_3,0}, \quad (3.62)$$

$$\tau_1(\pi_-(u_{K,P})) = \delta_{\tilde{K},0} \delta_{\tilde{P},0} \delta_{A_1-A_2+A_3+B_1-B_2+B_3,0}. \quad (3.63)$$

and

$$\tau_0(\pi_+(t_{K,P})) = \delta_{r_{K,P}^+,0} \sum_{n=0}^{\infty} (q_{K,P,n}^+ - \delta_{K,0} \delta_{P,0} \delta_{A_1+A_2-A_3+B_1+B_2-B_3,0}), \quad (3.64)$$

$$\tau_0(\pi_-(u_{K,P})) = \delta_{r_{K,P}^-,0} \sum_{n=0}^{\infty} (q_{K,P,n}^- - \delta_{\tilde{K},0} \delta_{\tilde{P},0} \delta_{A_1-A_2+A_3+B_1-B_2+B_3,0}). \quad (3.65)$$

(i) Equations (3.62) and (3.63) give $(\tau_1 \otimes \tau_1) r(\underline{A} \underline{A}')^0 = \delta_{A_1,-B_1} \delta_{A_2,0} \delta_{A_3,0} \delta_{B_2,0} \delta_{B_3,0} \lambda_{0,0}$. A computation of $v_{0,0}$ with $\delta_{A_1,-B_1} \delta_{A_2,0} \delta_{A_3,0} \delta_{B_2,0} \delta_{B_3,0} = 1$ gives the result.

(ii) Equations (3.62) and (3.65) yield

$$\begin{aligned} \tau_1(\pi_+(t_{K,P})) \tau_0(\pi_-(u_{K,P})) &= \delta_{K,0} \delta_{P,0} \delta_{A_2+B_2,A_3+B_3} \delta_{A_1,-B_1} \\ &\quad \times v_{\beta_1,\alpha'_1,\beta'_1}((\alpha_2 + \beta_2 + \alpha_3 + \beta_3)(\alpha'_1 + \beta'_1), A_3 + B_3). \end{aligned}$$

Equations (3.64) and (3.63) yield

$$\begin{aligned} \tau_0(\pi_+(t_{K,P})) \tau_1(\pi_-(u_{K,P})) &= \delta_{\tilde{K},0} \delta_{\tilde{P},0} \delta_{A_2+B_2,A_3+B_3} \delta_{A_1,-B_1} \\ &\quad \times v_{\beta_1,\alpha'_1,\beta'_1}((\alpha_2 + \beta_2 + \alpha_3 + \beta_3)(\alpha'_1 + \beta'_1), A_3 + B_3) \end{aligned}$$

and the result follows.

(iii) With (3.39) a direct computation gives

$$\tau_1(\pi_+(t_{K,P})) = \delta_{K,0} \delta_{P,0} \delta_{A_1+A_2-A_3+B_1+B_2-B_3,0}, \quad (3.66)$$

$$\tau_1(\pi_-(u_{K,P})) = \delta_{\tilde{K},0} \delta_{\tilde{P},0} \delta_{A_1-A_2+A_3+B_1-B_2+B_3,0}. \quad (3.67)$$

Using (3.66) and (3.67), $(\tau_1 \otimes \tau_1)(r(\underline{A} \underline{A}' \underline{A}'')^\circ) = \delta_{A_1,-B_1} \delta_{A_2,0} \delta_{A_3,0} \delta_{B_2,0} \delta_{B_3,0} v_{0,0}$. A computation of $v_{0,0}$ with $\delta_{A_1,-B_1} \delta_{A_2,0} \delta_{A_3,0} \delta_{B_2,0} \delta_{B_3,0} = 1$ gives the result.

(iv) We have $\delta(M_\beta^\alpha) M_{\beta'}^{\alpha'} = \delta(x) \delta(y) x' \delta(y') + x \delta^2(y) x' \delta(y')$ where x, x', y, y' are monomials ($\underline{\pi}$ omitted). Since

$$\underline{\pi}(x) = \sum_k \binom{\alpha}{k} a_{+,\alpha_1}^{\hat{k}_1} a_{-,\alpha_1}^{k_1} b_{+}^{\hat{k}_2} b_{-}^{k_2} b_{+}^{*\hat{k}_3} b_{-}^{*k_3} =: \sum_k \binom{\alpha}{k} c_k,$$

we get $\delta(\underline{\pi}(x)) = \sum_k g(k) \binom{\alpha}{k} c_k$.

Similarly, $\delta(\underline{\pi}(y)) = \sum_p g(p) \binom{\beta}{p} c_p$ and $\delta^2(\underline{\pi}(y)) = \sum_p g(p)^2 \binom{\beta}{p} c_p$.

Thus, with $c_{K,P} := c_k c_p c_{k'} c_{p'}$,

$$\delta(x) \delta(y) x' \delta(y') = \sum_{K,P} g(k) g(p) g(p') \binom{\alpha}{K} \binom{\beta}{P} c_{K,P},$$

$$x \delta^2(y) x' \delta(y') = \sum_{K,P} g(p)^2 g(p') \binom{\alpha}{K} \binom{\beta}{P} c_{K,P},$$

$$r(\delta(M_\beta^\alpha) M_{\beta'}^{\alpha'})^0 = \sum_{K,P} \delta_{h_{K,P},0} (g(k) + g(p)) g(p) g(p') \binom{\alpha}{K} \binom{\beta}{P} r(c_{K,P}) =: \sum_{K,P} \lambda_{K,P} r(c_{K,P}).$$

Since $r(c_k) = (-q)^{k_1} (-1)^{\alpha_2+\alpha_3} \pi_+(t_k) \otimes \pi_-(u_k)$ with t_k, u_k defined by

$$t_k := a_{\alpha_1}^{\hat{k}_1} b^{k_1} a^{\hat{k}_2} b^{k_2} a^{*\hat{k}_3} b^{k_3} \text{ and } u_k := a_{\alpha_1}^{\hat{k}_1} b^{k_1} b^{\hat{k}_2} a^{*k_2} b^{\hat{k}_3} a^{k_3},$$

we get

$$r(\delta(M_\beta^\alpha) M_{\beta'}^{\alpha'})^0 = \sum_{K,P} \lambda_{K,P} (-q)^{k_1+k'_1+p_1+p'_1} (-1)^{A_2+A_3+B_2+B_3} \pi_+(t_{K,P}) \otimes \pi_-(u_{K,P})$$

where $t_{K,P} = t_k t_p t_{k'} t_{p'}$ and $u_{K,P} = u_k u_p u_{k'} u_{p'}$. Direct computations yield

$$\tau_1(\pi_+(t_{K,P})) = \delta_{K,0} \delta_{P,0} \delta_{A_1+A_2-A_3+B_1+B_2-B_3,0},$$

$$\tau_1(\pi_-(u_{K,P})) = \delta_{\tilde{K},0} \delta_{\tilde{P},0} \delta_{A_1-A_2+A_3+B_1-B_2+B_3,0}.$$

The result follows.

(v) For the last equality, note that by (iv)

$$\oint \delta(A) A |\mathcal{D}|^{-3} = -2 \sum_{\alpha_1, \alpha'_1, \beta_1, \beta'_1} (\alpha'_1 + \beta'_1) \beta_1 \beta'_1 A_{\alpha_1 000}^{\beta_1 00} A_{\alpha'_1 00}^{\beta'_1 00} \delta_{\alpha_1 + \alpha'_1 + \beta_1 + \beta'_1, 0}.$$

The following change of variables $\alpha_1 \leftrightarrow \alpha'_1$, $\beta_1 \leftrightarrow \beta'_1$, implies by symmetry that this is equal to zero.

C. Proof of Lemma 3.4.16

(i) Following notations of Lemma 3.4.12, we have

$$M_\beta^\alpha J M_{\beta'}^{\alpha'} J^{-1} = \sum_{K,P} v_{K,P} c_{k,p} J c_{k',p'} J^{-1}$$

where $K = (k, k')$, $P = (p, p')$, $\lambda_{K,P} = g(p)g(p')v_k v_{k'} w_p w_{p'}$. Thus,

$$\tilde{\rho}(M_\beta^\alpha J M_{\beta'}^{\alpha'} J^{-1}) = (-1)^{A_2+A_3+B_2+B_3} \sum_{K,P} (-q)^{k_1+k'_1+p_1+p'_1} \lambda_{K,P} T_{K,P}^+ \otimes T_{K,P}^-$$

where $T_{K,P}^+ := \pi'_+(t_k t_p) \widehat{\pi}_+(t_{k'} t_{p'})$ and $T_{K,P}^- := \pi'_-(u_k u_p) \widehat{\pi}_-(u_{k'} u_{p'})$ with

$$\begin{aligned} t_k &:= a_{\alpha_1}^{\widehat{k}_1} b_{\alpha_1}^{*k_1} a^{\widehat{k}_2} b^{*k_2} a^{*\widehat{k}_3} b^{k_3}, \\ u_k &:= a_{\alpha_1}^{\widehat{k}_1} b_{\alpha_1}^{*k_1} b^{\widehat{k}_2} a^{*k_2} b^{*\widehat{k}_3} a^{k_3}. \end{aligned}$$

A direct computation leads to

$$\begin{aligned} \tau_1(T_{K,P}^+) &= \delta_{K,0} \delta_{P,0} \delta_{\alpha_1+\alpha_2-\alpha_3+\beta_1+\beta_2-\beta_3,0} \delta_{\alpha'_1+\alpha'_2-\alpha'_3+\beta'_1+\beta'_2-\beta'_3,0}, \\ \tau_1(T_{K,P}^-) &= \delta_{\widetilde{K},0} \delta_{\widetilde{P},0} \delta_{\alpha_1-\alpha_2+\alpha_3+\beta_1-\beta_2+\beta_3,0} \delta_{\alpha'_1-\alpha'_2+\alpha'_3+\beta'_1-\beta'_2+\beta'_3,0} \end{aligned}$$

which gives the result.

(ii) Using the commutation relations on \mathcal{A} , we see that there are real functions of (K, P) , denoted $\sigma_{K,P}^t$ and $\sigma_{K,P}^u$ such that

$$\begin{aligned} T_{K,P}^+ &= q^{\sigma_{K,P}^t} \pi'_+(t_{k,p}) \widehat{\pi}_+(t_{k',p'}), \\ T_{K,P}^- &= q^{\sigma_{K,P}^u} \pi'_-(u_{k,p}) \widehat{\pi}_-(u_{k',p'}), \\ t_{k,p} &:= a_{\alpha_1}^{\widehat{k}_1} a^{\widehat{k}_2} a^{*\widehat{k}_3} a_{\beta_1}^{\widehat{p}_1} a^{\widehat{p}_2} a^{*\widehat{p}_3} b_{\alpha_1}^{*k_1} b_{\beta_1}^{*p_1} b^{*k_2+p_2} b^{k_3+p_3}, \\ u_{k,p} &:= a_{\alpha_1}^{\widehat{k}_1} a^{*k_2} a^{k_3} a_{\beta_1}^{\widehat{p}_1} a^{*p_2} a^{p_3} b_{\alpha_1}^{*k_1} b_{\beta_1}^{*p_1} b^{\widehat{k}_2+\widehat{p}_2} b^{*\widehat{k}_3+\widehat{p}_3}. \end{aligned}$$

We have, under the hypothesis $\tau_1(T_{K,P}^-) = 1$,

$$\begin{aligned} \widehat{\pi}_+(t_{k',p'}) \varepsilon_{m,2j} &= (-1)^{\lambda'} q^{(2j-m)\lambda'} q_{2j-m-s+\beta'_1, |\alpha'_1|}^{\uparrow \alpha'_1} q_{2j-m-s, |\beta'_1|}^{\uparrow \beta'_1} \varepsilon_{m+s, 2j}, \\ s &:= -\alpha'_2 + \alpha'_3 - \beta'_2 + \beta'_3 = \alpha'_1 + \beta'_1, \\ \lambda' &:= \alpha'_2 + \alpha'_3 + \beta'_2 + \beta'_3, \\ \lambda &:= \alpha_2 + \alpha_3 + \beta_2 + \beta_3 \\ \tau_1(T_{K,P}^+) &= \delta_{\lambda,0} \delta_{\lambda',0}. \end{aligned}$$

and then,

$$\begin{aligned} (T_{K,P}^+)_{m,2j} &= q^{\sigma_{K,P}^t + s\lambda} (-1)^{\lambda'} q^{(2j-m)\lambda' + m\lambda} F_m F'_{2j-m} \delta_{A_1+B_1,0}, \\ F'_{2j-m} &:= q_{2j-m-\alpha'_1, |\alpha'_1|}^{\uparrow \alpha'_1} q_{2j-m-\alpha'_1-\beta'_1, |\beta'_1|}^{\uparrow \beta'_1}, \\ F_m &:= q_{m-\alpha_1, |\alpha_1|}^{\uparrow \alpha_1} q_{m-\beta_1-\alpha_1, |\beta_1|}^{\uparrow \beta_1}. \end{aligned}$$

Following the proof of Lemma 3.4.7, we see that $\tau_0(T_{K,P}^+)$ is possibly nonzero only in the two cases $\lambda' = 0$ or $\lambda = 0$.

Suppose first $\lambda = \lambda' = 0$. In that case, we have

$$\tau_0(T_{K,P}^+) = \lim_{2j \rightarrow \infty} \sum_{m=0}^{2j} ((q_{m,|\beta_1|}^{\uparrow_{\beta_1}} q_{2j-m,|\beta'_1|}^{\uparrow_{\beta'_1}})^2 - 1) = \sum_{m=0}^{\infty} ((q_{m,|\beta_1|}^{\uparrow_{\beta_1}})^2 - 1) + \sum_{m=0}^{\infty} ((q_{m,|\beta'_1|}^{\uparrow_{\beta'_1}})^2 - 1)$$

where the second equality comes from Lemma 3.4.17.

In the case $(\lambda = 0, \lambda' > 0)$, we get $\alpha'_1 = -\beta'_1$ and thus,

$$(T_{K,P}^+)_{m,2j} = q^{\sigma_{K,P}^t} q^{m\lambda} (q_{m,|\beta_1|}^{\uparrow_{\beta_1}} q_{2j-m,|\beta'_1|}^{\uparrow_{\beta'_1}})^2 \delta_{\alpha_1+\beta_1,0}.$$

Let us note $U_{2j} = \sum_{m=0}^{2j} q^{m\lambda} (q_{m,|\beta_1|}^{\uparrow_{\beta_1}} q_{2j-m,|\beta'_1|}^{\uparrow_{\beta'_1}})^2$ and $L_{2j} = \sum_{m=0}^{2j} q^{m\lambda} (q_{m,|\beta_1|}^{\uparrow_{\beta_1}})^2$.

Suppose $\beta'_1 > 0$. Since $(q_{2j-m,|\beta'_1|}^{\uparrow_{\beta'_1}})^2 - 1 = \sum_{|p|_1 \neq 0, p_i \in \{0,1\}} (-1)^{|p|_1} q^{r_p} q^{2(2j-m)|p|_1}$ where we have $r_p = 2 + \dots + 2\beta'_1$. As in the proof of Lemma 3.4.7 (ii), we can conclude that $U_{2j} - L_{2j}$ converges to 0. The case $\beta'_1 \leq 0$ is similar.

In the other case $(\lambda > 0, \lambda' = 0)$, the arguments are the same, replacing λ by λ' and α_1, β_1 by α'_1, β'_1 . Finally,

$$\begin{aligned} \tau_0(T_{K,P}^+) \tau_1(T_{K,P}^-) &= \delta_{\tilde{K},0} \delta_{\tilde{P},0} \delta_{\alpha_1,-\beta_1} \delta_{\alpha'_1,-\beta'_1} (\delta_{\lambda',0} \delta_{\alpha_2+\beta_2,\alpha_3+\beta_3} s_{\alpha,\beta} + \delta_{\lambda,0} \delta_{\alpha'_2+\beta'_2,\alpha'_3+\beta'_3} s_{\alpha',\beta'}), \\ s_{\alpha\beta} &:= q^{\beta_1(\alpha_3-\alpha_2)} \sum_{m=0}^{\infty} (q^{m\lambda} (q_{m,|\beta_1|}^{\uparrow_{\beta_1}})^2 - \delta_{\lambda,0}). \end{aligned}$$

A similar computation of $\tau_0(T_{K,P}^-)$ can be done following the same arguments. We find eventually

$$\tau_1(T_{K,P}^+) \tau_0(T_{K,P}^-) = \delta_{K,0} \delta_{P,0} \delta_{\alpha_1,-\beta_1} \delta_{\alpha'_1,-\beta'_1} (\delta_{\lambda',0} \delta_{\alpha_2+\beta_2,\alpha_3+\beta_3} s_{\alpha,\beta} + \delta_{\lambda,0} \delta_{\alpha'_2+\beta'_2,\alpha'_3+\beta'_3} s_{\alpha',\beta'})$$

and the result follows.

(iii) The same arguments of (i) apply here with minor changes.

(iv) follows from a slight modification of the proof of Lemma 3.4.14 (iv).

(v) is a straightforward consequence of (i, ii, iii, iv).

D. Proof of Lemma 3.4.17

We give a proof for β and $\beta' > 0$, the other cases being similar.

Since $(q_{m,|\beta|}^{\uparrow_{\beta}})^2 = \sum_{p_i \in \{0,1\}} (-1)^{|p|_1} q^{r_p} q^{2|p|_1 m}$ where $p = (p_1, \dots, p_{\beta})$ and $r_p := 2(p_1 + \dots + \beta p_{\beta})$, we get, with the notations $\lambda_{p,p'} := (-1)^{|p+p'|_1} q^{r_p+r_{p'}}$ and $U_{2j} := \sum_{m=0}^{2j} (q_{m,|\beta|}^{\uparrow_{\beta}} q_{2j-m,|\beta'|}^{\uparrow_{\beta'}})^2 - 1$,

$$\begin{aligned} U_{2j} &= \sum_{m=0}^{2j} \sum_{|p+p'|_1 > 0} \lambda_{p,p'} q^{2|p|_1 m + 2|p'|_1 (2j-m)} \\ &= \sum_{|p|_1 \geq |p'|_1, |p|_1 > 0} \lambda_{p,p'} V_{2j,p,p'} + \sum_{|p|_1 < |p'|_1, |p'|_1 > 0} \lambda_{p,p'} V'_{2j,p,p'} \end{aligned}$$

where

$$V_{2j,p,p'} = q^{4j|p'|_1} \sum_{m=0}^{2j} q^{2(|p|_1 - |p'|_1)m}, \quad V'_{2j,p,p'} = q^{4j|p|_1} \sum_{m=0}^{2j} q^{2(|p'|_1 - |p|_1)m}.$$

It is clear that $V_{2j,p,p'}$ has 0 for limit when $j \rightarrow \infty$ when $|p'|_1 > 0$, and $V'_{2j,p,p'}$ has 0 for limit when $j \rightarrow \infty$ when $|p|_1 > 0$. As a consequence,

$$U_{2j} = \sum_{|p|_1 > 0} \lambda_{p,0} V_{2j,p,0} + \sum_{|p'|_1 > 0} \lambda_{0,p'} V'_{2j,0,p'} + o(1).$$

The result follows as

$$\sum_{m=0}^{2j} ((q_{m,|\beta|}^{\uparrow\beta})^2 - 1) = \sum_{|p|_1 > 0} \lambda_{p,0} V_{2j,p,0} \text{ and } \sum_{m=0}^{2j} ((q_{m,|\beta'|}^{\uparrow\beta'})^2 - 1) = \sum_{|p'|_1 > 0} \lambda_{0,p'} V'_{2j,0,p'}.$$

Chapitre 4

Tadpoles and commutative spectral triples

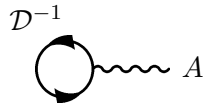
4.1 Introduction

The history of the noncommutative residue is now rather long [89], so we sketch it only briefly: after some approaches by Adler [2] and Manin [103] on the Korteweg-de Vries equation using a trace on the algebra of formal pseudodifferential operators in one dimension, and of Guillemin with his "soft" proof of Weyl's law on the eigenvalues of an elliptic operator [72], the noncommutative residue in any dimension was essentially initiated par Wodzicki in his thesis [150]. This residue gives, up to a multiplicative factor, the unique non-trivial trace on the algebra of pseudodifferential operators. Then, a link between this residue and the Dixmier trace was given by Connes in [25]. Thanks to Connes again [28,29], the setting of classical pseudodifferential operators on Riemannian manifolds without boundary was extended to a noncommutative geometry where the manifold is replaced by a not necessarily commutative algebra \mathcal{A} plus a Dirac-like operator \mathcal{D} via the notion of spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ where \mathcal{H} is the Hilbert space acted upon by \mathcal{A} and \mathcal{D} . The previous Dixmier trace is extended to the algebra of pseudodifferential operators naturally associated to the triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. This spectral point of view appears quite natural in the general framework of noncommutative geometry which goes beyond Riemannian geometry. From a physicist's point of view, this framework has many advantages: the spectral approach is motivated by quantum physics but not only since classical observables and infinitesimals are now on the same footing and even Dixmier's trace is related to renormalization. It is amazing to observe that most of classical geometrical notions like those defined in relativity or particle physics can be extended in this really noncommutative setting. Among others, some physical actions still make sense as in [25] where Dixmier's trace is used to compute the Yang–Mills action in the context of noncommutative differential geometry. Another example is the Einstein–Hilbert action: on a compact spin Riemannian 4-manifold, $\oint \mathcal{D}^{-2}$ coincides (up to a universal scalar) with the Einstein–Hilbert action, where \oint is precisely the noncommutative residue, a point first noticed by Connes; then, there was a brute force proof [90] and generalization [88] (see also [1]) of this fact which is particularly relevant here.

Since then, the case of compact manifolds with boundary has been studied, making clearer the links between noncommutative residues, Dixmier's trace and heat kernel expansion. This was achieved using Boutet de Monvel's algebra [55,71,124], in the case of conical singularities

[99, 123] or when the symbols are log-polyhomogeneous [98]. Besides, there are some applications of noncommutative residues for such manifolds to classical gravity [147] and to the unification of gravity with fundamental interactions [23]. Needless to say that in field theory, the one-loop divergencies, anomalies and different asymptotics of the effective action are directly obtained from the heat kernel method [141], so all of the above quoted mathematical results have profound applications to physics.

We are interested in possible cancellation of terms in the Chamseddine–Connes spectral action formula (1.6). We focus essentially on commutative spectral triples, for which we show that there are no tadpoles (see Definition 1.4.3). In particular, terms like $\oint A \mathcal{D}^{-1}$ are zero: in field theory, \mathcal{D}^{-1} is the Feynman propagator and $A \mathcal{D}^{-1}$ is a one-loop graph with fermionic internal line and only one external bosonic line A looking like a tadpole. More generally, the tadpoles are the A -linear terms in (1.6).



In [114], computations of $\oint |\mathcal{D}|^{-k}$ for some values of k are presented and formula like (1.11) also appear in [100] in the context of pseudodifferential elliptic operators. As a starting point, we investigate in section 4.2 the existence of tadpoles for manifolds with boundaries, considering following Chamseddine and Connes [23] the case of a chiral boundary condition on the Dirac operator. One of their original motivations was to show that the first two terms in the spectral action come with the right ratio and sign for their coefficients as in the modified Euclidean action used in gravitation. We generalize this approach to the perturbed Dirac operator by an internal fluctuation, ending up with no tadpoles up to order 5.

However, this approach stems from explicit computations of first heat kernel coefficients, so we cannot conclude that other integrals of the same type as tadpoles are zero. It is then natural to restrict to manifolds without boundary via a different method.

After some useful facts using the link between, we conclude in section 4.3, and using the results of section 1.4 and pseudodifferential techniques, that a lot of terms in (1.6) are zero.

4.2 Tadpoles and compact spin manifolds with boundary

Let M be a smooth compact Riemannian d -dimensional manifold with smooth boundary ∂M and let V be a given smooth vector bundle on M . We denote dx (resp. dy) the Riemannian volume form on M (resp. on ∂M).

Recall that a differential operator P is of Laplace type if it has locally the form

$$P = -(g^{\mu\nu} \partial_\mu \partial_\nu + \mathbb{A}^\mu \partial_\mu + \mathbb{B}) \quad (4.1)$$

where $(g^{\mu\nu})_{1 \leq \mu, \nu \leq d}$ is the inverse matrix associated to the metric g on M , and \mathbb{A}^μ and \mathbb{B} are smooth $L(V)$ -sections on M (endomorphisms). A differential operator D is of Dirac type if D^2 is of Laplace type, or equivalently if it has locally the following form

$$D = -i\gamma^\mu \partial_\mu + \phi$$

where $(\gamma^\mu)_{1 \leq \mu \leq d}$ gives V a Clifford module structure: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \text{Id}_V$, $(\gamma^\mu)^* = \gamma^\mu$.

A particular case of Dirac operator is given by the following formula

$$D = -i\gamma^\mu(\partial_\mu + \omega_\mu) \quad (4.2)$$

where the ω_μ are in $C^\infty(L(V))$.

If P is a Laplace type operator of the form (4.1), then (see [65, Lemma 1.2.1]) there is an unique connection ∇ on V and an unique endomorphism E such that $P = L(\nabla, E)$ where by definition

$$L(\nabla, E) := -(\text{Tr}_g \nabla^2 + E), \quad \nabla^2(X, Y) := [\nabla_X, \nabla_Y] - \nabla_{\nabla_X Y},$$

X, Y are vector fields on M and ∇^g is the Levi-Civita connection on M . Locally

$$\text{Tr}_g \nabla^2 := g^{\mu\nu}(\nabla_\mu \nabla_\nu - \Gamma_{\mu\nu}^\rho \nabla_\rho)$$

where $\Gamma_{\mu\nu}^\rho$ are the Christoffel coefficients of ∇^g . Moreover (with local frames of T^*M and V), $\nabla = dx^\mu \otimes (\partial_\mu + \omega_\mu)$ and E are related to $g^{\mu\nu}$, \mathbb{A}^μ and \mathbb{B} through

$$\omega_\nu = \frac{1}{2}g_{\nu\mu}(\mathbb{A}^\mu + g^{\sigma\varepsilon}\Gamma_{\sigma\varepsilon}^\mu \text{Id}), \quad (4.3)$$

$$E = \mathbb{B} - g^{\nu\mu}(\partial_\nu \omega_\mu + \omega_\nu \omega_\mu - \omega_\sigma \Gamma_{\nu\mu}^\sigma). \quad (4.4)$$

Suppose that $P = L(\nabla, E)$ is a Laplace type operator on M , and assume that χ is an endomorphism of $V_{\partial M}$ so that $\chi^2 = \text{Id}_V$. We extend χ on a collar neighborhood \mathcal{C} of ∂M in M with the condition $\nabla_d(\chi) = 0$ where the d^{th} -coordinate here is the radial coordinate (the geodesic distance of a point in M to the boundary ∂M).

Let $V_\pm := \Pi_\pm V$ be the sub-bundles of V on \mathcal{C} where $\Pi_\pm := \frac{1}{2}(\text{Id}_V \pm \chi)$ are the projections on the ± 1 eigenvalues of χ . We also fix an auxiliary endomorphism S on $V_{+\partial M}$ extended to \mathcal{C} .

This allows to define the mixed boundary operator $\mathcal{B} = \mathcal{B}(\chi, S)$ as

$$\mathcal{B}s := \Pi_+(\nabla_d + S)\Pi_+s|_{\partial M} \oplus \Pi_-s|_{\partial M}, \quad s \in C^\infty(V). \quad (4.5)$$

These boundary conditions generalize Dirichlet ($\Pi_- = \text{Id}_V$) and Neumann–Robin ($\Pi_+ = \text{Id}_V$) conditions.

We define $P_{\mathcal{B}}$ as the realization of P on \mathcal{B} , that is to say the closure of P defined on the space of smooth sections of V satisfying the boundary condition $\mathcal{B}s = 0$.

We are interested in the behavior of the heat kernel coefficients a_{d-n} defined through its expansion as $\Lambda \rightarrow \infty$ (see [65, Theorem 1.4.5])

$$\text{Tr}(e^{-\Lambda^{-2}D_{\mathcal{B}}^2}) \sim \sum_{n \geq 0} \Lambda^{d-n} a_{d-n}(D, \mathcal{B})$$

where D is a self-adjoint Dirac type operator. Moreover, we will use a perturbation $D \rightarrow D + A$, where A is a 1-form (a linear combination of terms of the type $f[D, g]$, where f and g are smooth functions on M). More precisely, we investigate the linear dependence of these coefficients with respect to A . It is clear that, since A is a differential operator of order 0, a perturbation $D \mapsto D + A$ transforms a Dirac type operator into another Dirac type operator.

This perturbation has consequences on the E and ∇ terms:

Lemma 4.2.1. *Let D be a Dirac type operator locally of the form (4.2) such that $\nabla_\mu := \partial_\mu + \omega_\mu$ is connection compatible with the Clifford action γ . Let A be a 1-form associated to D , so that A is locally of the form $-i\gamma^\mu a_\mu$ with $a_\mu \in C^\infty(U)$, (U, x_μ) being a local coordinate frame on M . Then $(D + A)^2 = L(\nabla^A, E^A)$ and $D^2 = L(\nabla, E)$ where,*

$$\begin{aligned}\omega_\mu^A &= \omega_\mu + a_\mu, \text{ thus } \nabla_\mu^A = \nabla_\mu + a_\mu \text{Id}_V, \\ E^A &= E + \frac{1}{4}[\gamma^\mu, \gamma^\nu]F_{\mu\nu}, \quad E = \frac{1}{2}\gamma^\mu\gamma^\nu[\nabla_\mu, \nabla_\nu], \quad F_{\mu\nu} := \partial_\mu(a_\nu) - \partial_\nu(a_\mu)\end{aligned}$$

Moreover, the curvature of the connection ∇^A is $\Omega_{\mu\nu}^A = \Omega_{\mu\nu} + F_{\mu\nu}$, where $\Omega_{\mu\nu} = [\nabla_\mu, \nabla_\nu]$.

In particular $\text{Tr } E^A = \text{Tr } E$.

Proof. This is quoted in [141, equation (3.27)].

$(D + A)^2 = L(\nabla^A, E^A) := -g^{\mu\nu}(\nabla_\mu^A \nabla_\nu^A - \Gamma_{\mu\nu}^\rho \nabla_\rho^A) - E^A$ and we get with $\nabla_\mu^A := \nabla_\mu + a_\mu \text{Id}_V$:

$$\begin{aligned}-(D + A)^2 &= \gamma^\mu \nabla_\mu^A \gamma^\nu \nabla_\nu^A = \gamma^\mu [\nabla_\mu^A, \gamma^\nu] \nabla_\nu^A + \gamma^\mu \gamma^\nu \nabla_\mu^A \nabla_\nu^A \\ &= \gamma^\mu [\nabla_\mu, \gamma^\nu] \nabla_\nu^A + \frac{1}{2}(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) \nabla_\mu^A \nabla_\nu^A + \frac{1}{2}\gamma^\mu \gamma^\nu [\nabla_\mu^A, \nabla_\nu^A] \\ &= -\gamma^\mu \gamma^\rho \Gamma_{\mu\rho}^\nu \nabla_\nu^A + g^{\mu\nu} \nabla_\mu^A \nabla_\nu^A + \frac{1}{2}\gamma^\mu \gamma^\nu [\nabla_\mu + a_\mu \text{Id}_V, \nabla_\nu + a_\nu \text{Id}_V].\end{aligned}\tag{4.6}$$

Since $\Gamma_{\mu\nu}^\rho = \Gamma_{\nu\mu}^\rho$, we get by comparison,

$$\begin{aligned}E^A &= \frac{1}{2}\gamma^\mu \gamma^\nu [\nabla_\mu + a_\mu \text{Id}_V, \nabla_\nu + a_\nu \text{Id}_V] = \frac{1}{2}\gamma^\mu \gamma^\nu ([\nabla_\mu, \nabla_\nu] + \partial_\mu(a_\nu) - \partial_\nu(a_\mu)) \\ &= \frac{1}{2}\gamma^\mu \gamma^\nu [\nabla_\mu, \nabla_\nu] + \frac{1}{4}[\gamma^\mu, \gamma^\nu](\partial_\mu(a_\nu) - \partial_\nu(a_\mu)).\end{aligned}\quad \square$$

Remark that even if quadratic terms in A^2 appear in the local presentation of the perturbation $D^2 \rightarrow (D + A)^2$ (in the b term), these terms do not appear in the invariant formulation (∇, E) since they are hidden in $\nabla_\mu^A \nabla_\nu^A$ of (4.6).

In the following, D and A are fixed and satisfy the hypothesis of Lemma 4.2.1. Indices i, j, k , and l range from 1 through the dimension d of the manifold and index a local orthonormal frame $\{e_1, \dots, e_d\}$ for the tangent bundle. Roman indices a, b, c , range from 1 through $d - 1$ and index a local orthonormal frame for the tangent bundle of the boundary ∂M . The vector field e_d is chosen to be the inward-pointing unit normal vector field. Greek indices are associated to coordinate frames.

Let R_{ijkl} , $\rho_{ij} := R_{ikkj}$ and $\tau := \rho_{ii}$ be respectively the components of the Riemann tensor, Ricci tensor and scalar curvature of the Levi-Civita connection. Let $L_{ab} := (\nabla_{e_a} e_b, e_d)$ be the second fundamental form of the hypersurface ∂M in M . Let “;” denote multiple covariant differentiations with respect to ∇^A and “.” denote multiple covariant differentiations with respect to ∇ and the Levi-Civita connection of M .

We will look at a chiral boundary condition. This is a mixed boundary condition natural to consider in order to preserve the existence of chirality on M and its boundary ∂M which are compatible with the (selfadjoint) Clifford action: we assume that the operator χ is selfadjoint and satisfies the following relations:

$$\{\chi, \gamma^d\} = 0, \quad [\chi, \gamma^a] = 0, \quad \forall a \in \{1, \dots, d - 1\}.\tag{4.7}$$

This condition was shown in [23] a natural assumption to enforce the hermiticity of the realization of the Dirac operator. It is known [65, Lemma 1.5.3] that ellipticity is preserved.

Since γ^d is invertible, $\dim V_+ = \dim V_-$ and $\text{Tr } \chi = 0$.

For an even-dimensional oriented manifold, there is a natural candidate χ satisfying (4.7), namely

$$\chi := \chi_{\partial M} = (-i)^{d/2-1} \gamma(e_1) \cdots \gamma(e_{d-1}).$$

This notation is compatible with (4.9). Recall that

$$\text{Tr}(\gamma^{i_1} \cdots \gamma^{i_{2k+1}}) = 0, \quad \forall k \in \mathbb{N}, \quad \text{Tr}(\gamma^i \gamma^j) = \dim V \delta^{ij}. \quad (4.8)$$

The natural realization of this boundary condition for the Dirac type operator $D + A$ is the operator $(D + A)_\chi$ which acts as $D + A$ on the domain $\{s \in C^\infty(V) : \Pi_- s|_{\partial M} = 0\}$. It turns out (see [12, Lemma 7]) that the natural boundary operator \mathcal{B}_χ^A defined by

$$\mathcal{B}_\chi^A s := \Pi_- (D + A) s|_{\partial M} \oplus \Pi_- s|_{\partial M}$$

is a boundary operator of the form (4.5) provided that $S = \frac{1}{2} \Pi_+ (-i[\gamma^d, A] - L_{aa} \chi) \Pi_+$.

Lemma 4.2.2. *Actually, S and $\chi_{;a}$ are independent of the perturbation A :*

$$(i) \quad S = -\frac{1}{2} L_{aa} \Pi_+.$$

$$(ii) \quad \chi_{;a} = \chi_{;a}.$$

Proof. (i) Since A is locally of the form $-i\gamma^j a_j$ with $a_j \in C^\infty(U)$, we obtain from (4.7),

$$\chi[\gamma^d, A] = -ia_j \chi[\gamma^d, \gamma^j] = -i \sum_{j < d} a_j \chi[\gamma^d, \gamma^j] = i \sum_{j < d} a_j [\gamma^d, \gamma^j] \chi = -[\gamma^d, A] \chi$$

and the result as a consequence of $\Pi_+ [\gamma^d, A] = [\gamma^d, A] \Pi_-$ and $\Pi_+ \Pi_- = 0$.

(ii) We have $\nabla_i^A = \nabla_i + a_i \text{Id}_V$ where $A =: -i\gamma^j a_j$, and since $(\nabla_i^A \chi)s = \nabla_i^A(\chi s) - \chi(\nabla_i^A s)$ for any $s \in C^\infty(V)$, using Lemma 4.2.1, $\nabla_i^A(\chi) = [\nabla_i + a_i \text{Id}_V, \chi] = [\nabla_i, \chi] = \nabla_i(\chi)$. \square

While S is not sensitive to the perturbation A , the boundary operator \mathcal{B}_χ^A depends a priori on A . We shall denote \mathcal{B}_χ the boundary operator \mathcal{B}_χ^A when $A = 0$.

The coefficients a_{d-k} for $0 \leq k \leq 4$ have been computed in [11] for general mixed boundary conditions in the case of Laplace type operators and in [12, Lemma 8] for Dirac type operators with chiral boundary conditions. We recall here these coefficients in our setting:

Proposition 4.2.3.

$$\begin{aligned} a_d(D + A, \mathcal{B}_\chi^A) &= (4\pi)^{-d/2} \int_M \text{Tr}_V 1 \, dx, \\ a_{d-1}(D + A, \mathcal{B}_\chi^A) &= 0, \\ a_{d-2}(D + A, \mathcal{B}_\chi^A) &= \frac{(4\pi)^{-d/2}}{6} \left\{ \int_M \text{Tr}_V (6E^A + \tau) \, dx + \int_{\partial M} \text{Tr}_V (2L_{aa} + 12S) \, dy \right\}, \\ a_{d-3}(D + A, \mathcal{B}_\chi^A) &= \frac{(4\pi)^{-(d-1)/2}}{384} \int_{\partial M} \text{Tr}_V \{ 96\chi E^A + 3L_{aa}^2 + 6L_{ab}^2 + 96SL_{aa} + 192S^2 - 12\chi_{;a}^2 \} \, dy, \\ a_{d-4}(D + A, \mathcal{B}_\chi^A) &= \frac{(4\pi)^{-d/2}}{360} \left\{ \int_M \text{Tr}_V \{ 60\tau E^A + 180(E^A)^2 + 30(\Omega_{ij}^A)^2 + 5\tau^2 - 2\rho^2 + 2R^2 \} \, dx \right. \\ &\quad \left. + \int_{\partial M} \text{Tr}_V \{ 180\chi E_{;d}^A + 120E^A L_{aa} + 720SE^A + 60\chi\chi_{;a}\Omega_{ad}^A + T \} \, dy \right\}. \end{aligned}$$

where

$$T := 20\tau L_{aa} + 4R_{adad}L_{bb} - 12R_{adbda}L_{ab} + 4R_{abcb}L_{ac} + \frac{1}{21}(160L_{aa}^3 - 48L_{ab}^2L_{cc} + 272L_{ab}L_{bc}L_{ac} \\ + 120\tau S + 144SL_{aa}^2 + 48SL_{ab}^2 + 480(S^2L_{aa} + S^3) - 42\chi_{;a}^2L_{bb} + 6\chi_{;a}\chi_{;b}L_{ab} - 120\chi_{;a}^2S)$$

is independent of A .

The following proposition shows that there are no tadpoles up to order 5 in manifolds endowed with a chiral boundary condition.

Theorem 4.2.4. *Let M be an even d -dimensional compact oriented spin Riemannian manifold with smooth boundary ∂M and spin bundle V . Let $D := -i\gamma^j\nabla_j$ be the classical Dirac operator, and $\chi = \chi_{\partial M} = (-i)^{d/2-1}\gamma(e_1)\cdots\gamma(e_{d-1})$ where $(e_i)_{1\leq i\leq d}$ is a local orthonormal frame of TM .*

The perturbation $D \rightarrow D + A$ where $A = -i\gamma^ja_j$ is a 1-form for D , induces, under the chiral boundary condition, the following perturbations on the heat kernel coefficients where we set $c_{d-k}(A) := a_{d-k}(D + A, \mathcal{B}_\chi^A) - a_{d-k}(D, \mathcal{B}_\chi)$:

$$(i) \ c_d(A) = c_{d-1}(A) = c_{d-2}(A) = c_{d-3}(A) = 0.$$

$$(ii) \ c_{d-4}(A) = -\frac{1}{6(2\pi)^{d/2}} \int_M F_{\mu\nu} F^{\mu\nu} dx.$$

In other words, the coefficients a_{d-k} for $0 \leq k \leq 3$ are unperturbed, a_{d-4} is only perturbed by quadratic terms in A and there are no linear terms in A in $a_{d-k}(D + A, \mathcal{B}_\chi^A)$ for $k \leq 5$.

Remark 4.2.5. *When A is selfadjoint, all coefficients $a_{d-k}(D + A, \mathcal{B}_\chi^A)$ and $a_{d-k}(D, \mathcal{B}_\chi)$ are real while linear contributions in A are purely imaginary, modulo traces of γ and χ matrices and their covariant derivatives. Since the invariant terms appearing as integrands of \int_M and $\int_{\partial M}$ in the coefficients at higher order are polynomial in S , χ , R , E^A and Ω^A , and their covariant derivatives, one expects no linear terms in A at any order.*

Proof. (i) The fact that $c_d(A) = c_{d-1}(A) = 0$ follows from Proposition 4.2.3.

Since by Lemma 4.2.2, $c_{d-2}(A) = (4\pi)^{-d/2} \int_M \text{Tr}_V(E^A - E_A) dx$, we get $c_{d-2}(A) = 0$ because $\text{Tr}_V E^A = \text{Tr}_V E$ by Lemma 4.2.1.

From Proposition 4.2.3 and Lemma 4.2.2, we get $c_{d-3}(A) = \frac{1}{4}(4\pi)^{-(d-1)/2} \int_{\partial M} \text{Tr}_V \{ \chi(E^A - E) \}$.

Since $\chi(E^A - E) = (-i)^{d/2}\gamma^1\cdots\gamma^{d-1}[\gamma^j, \gamma^k]F_{jk}$, (4.8) yields $\text{Tr}_V \chi(E^A - E) = 0$ because d is even.

(ii) Since $\text{Tr}_V(E^A - E) = 0$ and $\text{Tr}_V \chi(E^A - E) = 0$, we obtain $\text{Tr}_V S(E^A - E) = 0$ from Lemma 4.2.2. Thus, using Proposition 4.2.3 and Lemma 4.2.2,

$$c_{d-4}(A) = \frac{(4\pi)^{-d/2}}{360} \left\{ \int_M \text{Tr}_V \{ 180((E^A)^2 - E^2) + 30((\Omega_{ij}^A)^2 - (\Omega_{ij})^2) \} dx \right. \\ \left. + \int_{\partial M} \text{Tr}_V \{ 180\chi(E_{;d}^A - E_{;d}) + 60\chi\chi_{;a}(\Omega_{ad}^A - \Omega_{ad}) \} dy \right\}.$$

We obtain locally $\text{Tr}_V ((E^A)^2 - E^2) = \frac{1}{16} \text{Tr}([\gamma^\mu, \gamma^\nu][\gamma^\rho, \gamma^\sigma])F_{\mu\nu}F_{\rho\sigma}$ using Lichnérowicz' formula $E = -\frac{1}{4}\tau$. Since $\text{Tr}_V([\gamma^\mu, \gamma^\nu][\gamma^\rho, \gamma^\sigma]) = 4.2^{d/2}(g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma})$,

$$\text{Tr}_V ((E^A)^2 - E^2) = -2^{d/2-1}F_{\mu\nu}F^{\mu\nu}.$$

∇ being the spin connection associated to the spin structure of M , we have $\Omega_{ij} = \frac{1}{4}\gamma^k\gamma^l R_{ijkl}$. So $R_{ijkl} = -R_{ijlk}$ implies $\text{Tr}_V \Omega_{ij} = 0$. Hence, with Lemma 4.2.1,

$$\text{Tr}_V ((\Omega_{ij}^A)^2 - \Omega_{ij}^2) = 2^{d/2} F_{ij}^2 = 2^{d/2} F_{\mu\nu} F^{\mu\nu}.$$

Moreover, $E_{;d}^A = [\nabla_d + a_d, E^A] = [\nabla_d, E + \frac{1}{4}[\gamma^i, \gamma^j]F_{ij}] = E_{;d} + \frac{1}{4}[\nabla_d, [\gamma^i, \gamma^j]]F_{ij}$.

Using $[\nabla_i, \gamma^i] = \gamma(\nabla_i e_i)$ and (4.8),

$$\text{Tr}_V (\chi(E_{;d}^A - E_{;d})) = (-i)^{d/2} \frac{1}{2} F_{ij} \text{Tr}_V \{ \gamma^1 \cdots \gamma^{d-1} (\gamma(\nabla_d e_i) \gamma^j + \gamma^i \gamma(\nabla_d e_j)) \} = 0.$$

It remains to check that $\text{Tr}_V (\chi \chi_{;a} (\Omega_{ad}^A - \Omega_{ad})) = 0$. Let $\chi_M = -i\chi\gamma^d$ be the grading operator (see (4.9)). Since χ_M commutes with the spin connection operator ∇ (see [68, p. 396]),

$$0 = [\nabla_a, \chi_M] = [\nabla_a, \chi\gamma^d] = \chi_{;a}\gamma^d + \chi[\nabla_a, \gamma^d] = \chi_{;a}\gamma^d + \chi\gamma(\nabla_a e_d)$$

and thus $\chi\chi_{;a} = -\gamma(\nabla_a e_d)\gamma^d = -\Gamma_{ad}^j \gamma^j \gamma^d$, where $\Gamma_{ad}^j = -\Gamma_{aj}^d$ since (e_j) is an orthonormal frame. So $\text{Tr}_V (\chi\chi_{;a}) = -\Gamma_{ad}^j \delta^{jd} = -\Gamma_{ad}^d = 0$. Finally, the result on c_{d-4} follows from Lemma 4.2.1 as $\text{Tr}_V (\chi\chi_{;a} (\Omega_{ad}^A - \Omega_{ad})) = \text{Tr}_V (\chi\chi_{;a}) F_{ad}$.

The coefficient $a_{d-5}(D + A, \mathcal{B}_\chi^A)$ is computed in [13]. One can check directly as above that linear terms in A are not present. The computation uses the fact that the trace of the following terms $\chi E_{;dd}^A$, $E_{;d}^A S$, $\chi(E^A)^2$, $E^A S^2$, $\chi_{;a}\chi_{;b}\Omega_{ab}^A$, $\chi_{;a}^2 E^A$, do not have linear terms in A . \square

In the following, we investigate the above conjecture with Chamseddine–Connes pseudodifferential calculus applied to compact spin manifolds without boundary and Riemannian spectral triples. We also see, using Wodzicki residue, how to compute some noncommutative integrals in this setting.

4.3 Commutative spectral triples

4.3.1 Commutative geometry

Definition 4.3.1. Consider a commutative spectral triple given by a compact Riemannian spin manifold M of dimension d without boundary and its Dirac operator \mathcal{D} associated to the Levi-Civita connection. This means $(\mathcal{A} := C^\infty(M), \mathcal{H} := L^2(M, S), \mathcal{D})$ where S is the spinor bundle over M . This triple is real since, due to the existence of a spin structure, the charge conjugation operator generates an anti-linear isometry J on \mathcal{H} such that

$$JaJ^{-1} = a^*, \quad \forall a \in \mathcal{A},$$

and when d is even, the grading is given by the chirality matrix

$$\chi_M := (-i)^{d/2} \gamma^1 \gamma^2 \cdots \gamma^d. \quad (4.9)$$

Such a triple is said to be a commutative geometry (see [33] and [34] for the role of J in the nuance between spin and spin^c manifold).

Since $JaJ^{-1} = a^*$ for $a \in \mathcal{A}$, we get that in a commutative geometry,

$$JAJ^{-1} = -\epsilon A^*, \quad \forall A \in \Omega_{\mathcal{D}}^1(\mathcal{A}). \quad (4.10)$$

4.3.2 No tadpoles

The appearance of tadpoles never occurs in commutative geometries, as quoted in [37, Lemma 1.145] for the dimension $d = 4$. This fact means that a given geometry $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a critical point for the spectral action (1.6).

Theorem 4.3.2. *There are no tadpoles on a commutative geometry, namely, for any one-form $A = A^* \in \Omega_{\mathcal{D}}^1(\mathcal{A})$, $\text{Tad}_{\mathcal{D}+A}(k) = 0$, for any $k \in \mathbb{Z}$, $k \leq d$.*

Proof. Since $\tilde{A} = 0$ when $A = A^*$ by (4.10), the result follows from Corollary 1.4.5. \square

There are similar results in the following

Lemma 4.3.3. *Under same hypothesis, for any $k, l \in \mathbb{N}$*

- (i) $\oint A \mathcal{D}^{-k} = -\epsilon^{k+1} \oint A \mathcal{D}^{-k}$,
- (ii) $\oint \chi A \mathcal{D}^{-k} = -\epsilon^{k+1} \oint \chi A \mathcal{D}^{-k}$,
- (iii) $\oint A^l |\mathcal{D}|^{-k} = (-\epsilon)^l \oint A^l |\mathcal{D}|^{-k}$,
- (iv) $\oint \chi A^l |\mathcal{D}|^{-k} = (-\epsilon)^l \oint \chi A^l |\mathcal{D}|^{-k}$.

Proof.

$$\begin{aligned} \oint A \mathcal{D}^{-k} &= \overline{\oint J A \mathcal{D}^{-k} J^{-1}} = \overline{\oint J A J^{-1} (\epsilon^k \mathcal{D}^{-k})} = -\epsilon^{k+1} \overline{\oint A^* \mathcal{D}^{-k}} = -\epsilon^{k+1} \oint \mathcal{D}^{-k} A \\ &= -\epsilon^{k+1} \oint A \mathcal{D}^{-k}. \end{aligned}$$

The same argument gives the other equalities using $\chi A = -A\chi$ and $\chi|\mathcal{D}| = |\mathcal{D}|\chi$. \square

Lemma 4.3.4. *For any one-form A , $\oint (A \mathcal{D}^{-1})^k = 0$ when $k \in \mathbb{N}$ is odd.*

Proof. We have

$$\begin{aligned} \oint (A \mathcal{D}^{-1})^k &= \overline{\oint J (A \mathcal{D}^{-1})^k J^{-1}} = \overline{\oint (J A J^{-1} J \mathcal{D}^{-1} J^{-1})^k} = (-1)^k \epsilon^{2k} \overline{\oint (A^* \mathcal{D}^{-1})^k} \\ &= (-1)^k \oint (A \mathcal{D}^{-1})^k \end{aligned} \quad (4.11)$$

(which shows again that $\oint A \mathcal{D}^{-1} = 0$). \square

4.3.3 Miscellaneous for commutative geometries

To show that more noncommutative integrals, where the use of the operator J in the trick (4.11) is not sufficient, are nevertheless zero, we need to use the Wodzicki residue (see [151, 152]): in a chosen coordinate system and local trivialization (x, ξ) of T^*M , this residue is

$$\text{wres}_x(X) := \int_{S_x^* M} \text{Tr}(\sigma_{-d}^X(x, \xi)) |d\xi| |dx^1 \wedge \cdots \wedge dx^d|, \quad (4.12)$$

where $\sigma_{-d}^X(x, \xi)$ is the symbol of the classical pseudodifferential operator X in the chosen coordinate frame (x_1, \dots, x_d) , which is homogeneous of degree $-d := -\dim(M)$ and taken at

point $(x, \xi) \in T^*(M)$, $d\xi$ is the normalized restriction of the volume form to the unit sphere $S_x^*M \simeq \mathbb{S}^{d-1}$, so we assume $d \geq 2$ to get S_x^*M connected.

This $wres_x(X)$ appears to be a one-density not depending on the local representation of the symbol (see [68, 152]), so

$$Wres(X) := \int_M wres_x(X) \quad (4.13)$$

is well defined.

The noncommutative integral \oint coincides with the Wodzicki residue, up to a scalar: since both \oint and $Wres$ are traces on the set of pseudodifferential operators, the uniqueness of the trace [152] gives the proportionality

$$\oint X = c_d Wres(X) \quad (4.14)$$

where c_d is a constant depending only on d . Computing separately $\oint |\mathcal{D}|^{-d}$ and $Wres(|\mathcal{D}|^{-d})$, we get $c_d > 0$. (Note that \oint is not a positive functional, see Lemma 4.3.18.)

Lemma 1.4.1 follows for instance from the fact that $\int_M wres_x(X^*) = \overline{\int_M wres_x(X)}$.

Note that $Wres$ is independent of the metric.

As noticed by Wodzicki, $\oint X$ is equal to -2 times the coefficient in $\log t$ of the asymptotics of $\text{Tr}(X e^{-t\mathcal{D}^2})$ as $t \rightarrow 0$. It is remarkable that this coefficient is independent of \mathcal{D} and this gives a close relation between the ζ function and the heat-kernel expansion with $Wres$. Actually, by [70, Theorem 2.7]

$$\text{Tr}(X e^{-t\mathcal{D}^2}) \sim_{t \rightarrow 0^+} \sum_{k=0}^{\infty} a_k t^{(j-\text{ord}(X)-d)/2} + \sum_{k=0}^{\infty} (-a'_k \log t + b_k) t^k, \quad (4.15)$$

so $\oint X = 2a'_0$. Since, via Mellin transform, $\text{Tr}(X \mathcal{D}^{-2s}) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}(X e^{-t\mathcal{D}^2}) dt$, the non-zero coefficient a'_k , $k \neq 0$ create a pole of $\text{Tr}(X \mathcal{D}^{-2s})$ of order $k+2$ since $\int_0^1 t^{s-1} \log(t)^k = \frac{(-1)^k k!}{s^{k+1}}$ and

$$\Gamma(s) = \frac{1}{s} + \gamma + s g(s) \quad (4.16)$$

where γ is the Euler constant and the function g is also holomorphic around zero.

We have $\oint 1 = 0$ and more generally, $Wres(P) = 0$ for all zero-order pseudodifferential projections [151].

For extension to log-polyhomogeneous pseudodifferential operators, see [98].

When M has a boundary, some a'_k are non zero, the dimension spectrum can be non simple (even if it is simple for the Dirac operator, see for instance [99]).

On a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, changing the product on \mathcal{A} may or not affect the dimension spectrum: for instance, there is no change when one goes from the commutative torus to the noncommutative one, while the dimension spectrum of $SU_q(2)$ which is bounded from below, does not coincide with the dimension spectrum of the sphere \mathbb{S}^3 corresponding to $q = 1$.

We first introduce few necessary notations. In the following we fix a local coordinate frame $(U, (x_i)_{1 \leq i \leq n})$ which is normal at $x_0 \in M$, and denote σ_k^X the k -homogeneous symbol of any

classical pseudodifferential operator X on M , in this local coordinate frame. The Dirac operator is locally of the form—compatible with (4.2)

$$\mathcal{D} = -i\gamma(dx^j) (\partial_{x^j} + \omega_j(x)) \quad (4.17)$$

where ω_j is the spin connection, γ is the Clifford multiplication of one-forms [68, page 392]. Here we make the choice of gauge given by $h := \sqrt{g}$ which gives [68, Exercise 9.6]

$$\omega_i = -\frac{1}{4} (\Gamma_{ij}^k g_{kl} - \partial_{x^j}(h_j^\alpha) \delta_{\alpha\beta} h_l^\beta) \gamma(dx^j) \gamma(dx^l), \quad \gamma(dx^j) = \sqrt{g^{-1}}^{jk} \gamma_k$$

where $\gamma^j = \gamma_j$ are the selfadjoint constant γ matrices satisfying $\{\gamma^i, \gamma^j\} = \delta^{ij}$. Thus

$$\sigma^{\mathcal{D}}(x, \xi) = \sqrt{g^{-1}}^{jk} \gamma_k (\xi_j - i\omega_j(x)).$$

We have chosen normal (or geodesic) coordinates around the base point x_0 . Since

$$\begin{aligned} g_{ij}(x) &= g_{ij}(x_0) + \frac{1}{3} R_{ijkl} x^k x^l + o(\|x\|^3), \\ g^{ij}(x) &= g^{ij}(x_0) - \frac{1}{3} R^i{}_k{}^j{}_l x^k x^l + o(\|x\|^3), \\ g_{ij}(x_0) &= \delta_{ij}, \quad \Gamma_{ij}^k(x_0) = 0, \end{aligned}$$

the matrices $h(x)$ and $h^{-1}(x)$ have no linear terms in x . Thus

$$\omega_i(x_0) = 0.$$

We could also have said that parallel translation of a basis of the cotangent bundle along the radial geodesics emanating from x_0 yields a trivialization (this is the radial gauge) such that $\omega_i(x_0) = 0$. In particular, using product formulae for symbols and the fact that in the decomposition $D = \mathcal{D} + P$, $P \in OP^{-\infty}$, we get for $k \in \mathbb{N}$

$$\sigma_1^{\mathcal{D}}(x, \xi) = \sqrt{g^{-1}}^{jk}(x) \gamma_k \xi_j = \gamma(\xi), \quad \sigma_1^{\mathcal{D}}(x_0, \xi) = \gamma^j \xi_j, \quad (4.18)$$

$$\sigma_0^{\mathcal{D}}(x, \xi) = -i\sqrt{g^{-1}}^{jk}(x) \gamma_k \omega_j(x), \quad \sigma_0^{\mathcal{D}}(x_0, \xi) = 0, \quad (4.19)$$

$$\partial_{x^k} \sigma_1^{\mathcal{D}}(x_0, \xi) = 0, \quad (4.20)$$

$$\sigma_{-1}^{\mathcal{D}^{-1}}(x, \xi) = \sqrt{g^{-1}}^{jk}(x) \gamma_j \xi_k \|\xi\|_x^{-2}, \quad \|\xi\|_x^2 := g^{jk}(x) \xi_j \xi_k \quad (4.21)$$

$$\partial_{x^k} \sigma_{-1}^{\mathcal{D}^{-1}}(x_0, \xi) = 0. \quad (4.22)$$

We will use freely the fact that the symbol of a one-form A can be written as

$$\sigma^A(x, \xi) = \sigma_0^A(x) = -i a_k(x) \gamma^k \quad (4.23)$$

with $a_k(x) \in i\mathbb{R}$ when $A = A^*$.

When d is even (so $\epsilon = 1$), remark that for $k = l$ and $A_i = a_i[\mathcal{D}, b_i]$ and $a = \prod_{i=1}^k a_i$, then by [38, page 231 (actually, χ is missing)], [113] or [68, p. 479] when $k = d$, (M is supposed to be oriented)

$$\oint \chi A_1 \cdots A_k |\mathcal{D}|^{-k} = c'_k \int_M \hat{A}(R)^{(d-k)} \wedge a db_1 \wedge \cdots \wedge db_k$$

where $\hat{A}(R)$ is the \hat{A} -genus associated to the Riemannian curvature R . Since we have $\hat{A}(R) \in \oplus_{j \in \mathbb{N}} \Omega^{4j}(M, \mathbb{R})$, $\int \chi A^k |D|^{-k}$ can be non zero only when $k = d - 4j$. For instance in dimension $d=2$, for $j=0$,

$$\sigma_{-2}^{\chi A_1 A_2 \mathcal{D}^{-2}}(x, \xi) = \sigma_0^{\chi A_1 A_2}(x) \sigma_{-2}^{\mathcal{D}^{-2}}(x, \xi) = -a_1(x) a_2(x) \chi g^{jk}(x) \gamma_j \gamma_k \frac{1}{g^{Im}(x) \xi_l \xi_m}.$$

Thus $wres_x(\chi A_1 A_2 \mathcal{D}^{-2}) = -2 a_1(x) a_2(x) \sqrt{\det g_x} \operatorname{Tr}(\chi \gamma^j \gamma^k)$, so if ν_g is the Riemannian density,

$$\int \chi A_1 A_2 \mathcal{D}^{-2} = -2c_d \operatorname{Tr}(\chi \gamma^j \gamma^k) \int_M a_1 a_2 \nu_g. \quad (4.24)$$

Actually, this last equality is nothing else than Wodzicki–Connes' trace theorem, see [68, section 7.6], and this is equal to $c'_d \int_M a_1 a_2 db_1 \wedge db_2$ as claimed above.

We introduce a few subspaces of the pseudodifferential operators space $\Psi(M)$. Let

$$\begin{aligned} \mathcal{B}_e &:= \{ P \in \Psi(M) : \sigma_j^P \in E_j, \forall j \in \mathbb{Z} \} \quad \text{e for even,} \\ \mathcal{B}_o &:= \{ P \in \Psi(M) : \sigma_j^P \in O_j, \forall j \in \mathbb{Z} \} \quad \text{o for odd,} \end{aligned}$$

such that, for $m = 2^{[d/2]}$,

$$\begin{aligned} E_j &:= \{ f \in C^\infty(U \times \mathbb{R}^d \setminus \{0\}, \mathcal{M}_m(\mathbb{C})) : f(x, \xi) = \sum_{i \in I} \frac{\xi^{\beta^i}}{\|\xi\|_x^{2k_i}} h_i(x), \quad I \neq \emptyset, \\ &\quad k_i \in \mathbb{N}, \beta^i \in \mathbb{N}^d, |\beta^i| - 2k_i = j, h_i \in C^\infty(U, \mathcal{M}_m(\mathbb{C})) \}, \\ O_j &:= \{ f \in C^\infty(U \times \mathbb{R}^d \setminus \{0\}, \mathcal{M}_m(\mathbb{C})) : f(x, \xi) = \sum_{i \in I} \frac{\xi^{\beta^i}}{\|\xi\|_x^{2k_i+1}} h_i(x), \quad I \neq \emptyset, \\ &\quad k_i \in \mathbb{N}, \beta^i \in \mathbb{N}^d, |\beta^i| - (2k_i + 1) = j, h_i \in C^\infty(U, \mathcal{M}_m(\mathbb{C})) \}. \end{aligned}$$

Lemma 4.3.5. *For any $j, j' \in \mathbb{Z}$ and $\alpha \in \mathbb{N}^d$,*

- (i) $E_j E_{j'} \subseteq E_{j+j'}$ and $\partial_\xi^\alpha E_j \subseteq E_{j-|\alpha|}$, $\partial_x^\alpha E_j \subseteq E_j$.
- (ii) $O_j O_{j'} \subseteq E_{j+j'}$ and $\partial_\xi^\alpha O_j \subseteq O_{j-|\alpha|}$, $\partial_x^\alpha O_j \subseteq O_j$.
- (iii) $O_j E_{j'}$ and $E_{j'} O_j$ are included in $O_{j+j'}$.
- (iv) \mathcal{B}_e is a sub-algebra of $\Psi(M)$.
- (v) $\mathcal{B}_e \mathcal{B}_e, \mathcal{B}_o \mathcal{B}_o$ are included in \mathcal{B}_e , and $\mathcal{B}_e \mathcal{B}_o, \mathcal{B}_o \mathcal{B}_e$ are included in \mathcal{B}_o .

Proof. (i) Let $f \in E_j$ and $\alpha \in \mathbb{N}^d$. We have, if $f(x, \xi) = \sum_{i \in I} \frac{\xi^{\beta^i}}{\|\xi\|_x^{2k_i}} h_i(x)$,

$$\partial_\xi^\alpha f = \sum_{i \in I} \partial_\xi^\alpha \left(\frac{\xi^{\beta^i}}{\|\xi\|_x^{2k_i}} \right) h_i(x) = \sum_{i \in I} \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \partial_\xi^{\alpha-\gamma} (\xi^{\beta^i}) \partial_\xi^\gamma \left(\frac{1}{\|\xi\|_x^{2k_i}} \right) h_i(x).$$

We check by induction that we can write

$$\partial_\xi^\gamma \left(\frac{1}{\|\xi\|_x^{2k_i}} \right) = \frac{1}{\|\xi\|_x^{2k_i(|\gamma|+1)}} \sum_p \lambda_p \prod_{j=1}^{|\gamma|} \partial_\xi^{\beta^{j,p}} \|\xi\|_x^{2k_i}$$

where λ_p are real numbers, the sum on indices p is finite, and $\sum_{j=1}^{|\gamma|} \beta^{j,p} = \gamma$. As a consequence, since $\|\xi\|_x^{2k_i} = (g^{kl}(x)\xi_k\xi_l)^{k_i}$ is a homogeneous polynomial in ξ of degree $2k_i$, we get $\partial_\xi^\alpha f \in E_{j-|\alpha|}$. The inclusions $E_j E_{j'} \subseteq E_{j+j'}$, $\partial_x^\alpha E_j \subseteq E_j$ are straightforward.

(ii) The proof is similar to (i) since by induction

$$\partial_\xi^\gamma \left(\frac{1}{\|\xi\|_x} \right) = \frac{1}{\|\xi\|_x^{2|\gamma|+1}} \sum_p \lambda_p \prod_{j=1}^{|\gamma|} \partial_\xi^{\beta^{j,p}} \|\xi\|_x^2$$

where λ_p are real numbers, the sum on the indices p is finite and $\sum_{j=1}^{|\gamma|} \beta^{j,p} = \gamma$.

(iii) Straightforward.

(iv) The product symbol formula for two classical pseudodifferential operators $P \in \Psi^p(M)$, $Q \in \Psi^q(M)$ gives

$$\sigma_{p+q-j}^{PQ} = \sum_{\alpha \in \mathbb{N}^d} \sum_{k \geq 0, |\alpha|+k \leq j} i^{|\alpha|} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \sigma_{p-j+|\alpha|+k}^P \partial_x^\alpha \sigma_{q-k}^Q. \quad (4.25)$$

The presence of the factor $i^{|\alpha|}$ that will be crucial in later arguments like Lemma 4.3.10.

If $P, Q \in \mathcal{B}_e$, we see that by (i), $\partial_\xi^\alpha \sigma_{p-j+|\alpha|+k}^P \in E_{p-j+k}$ and $\partial_x^\alpha \sigma_{q-k}^Q \in E_{q-k}$. Again by (i), we obtain $\partial_\xi^\alpha \sigma_{p-j+|\alpha|+k}^P \partial_x^\alpha \sigma_{q-k}^Q \in E_{p+q-j}$, so the result follows from (4.25).

(v) A similar argument as (iv) can be applied, using (ii) to obtain $\mathcal{B}_o \mathcal{B}_o \subseteq \mathcal{B}_e$ and (iii) to get $\mathcal{B}_o \mathcal{B}_e \subseteq \mathcal{B}_o$, $\mathcal{B}_e \mathcal{B}_o \subseteq \mathcal{B}_o$. \square

\mathcal{B}_e and \mathcal{B}_o are stable by inverse:

Lemma 4.3.6. *Let $P \in \mathcal{B}_e$ (resp. \mathcal{B}_o) be an elliptic classical pseudodifferential operator in $\Psi^p(M)$ with $\sigma_p^P(x, \xi) = \|\xi\|_x^p$, $p \in \mathbb{N}$. Then any parametrix P^{-1} of P is in \mathcal{B}_e (resp. \mathcal{B}_o).*

Proof. Assume $P \in \mathcal{B}_e$ so p is even. From the parametrix equation $PP^{-1} = 1$, we obtain $\sigma_{-p}^{P^{-1}} = (\sigma_p^P)^{-1} = \|\xi\|_x^{-p} \in E_{-p}$. Moreover, using (4.25), we see that for any $j \in \mathbb{N}^*$,

$$\sigma_{-p-j}^{P^{-1}} = -(\sigma_p^P)^{-1} \left(\sum_{0 \leq k < j} \sigma_{p-j+k}^P \sigma_{-p-k}^{P^{-1}} + \sum_{0 < |\alpha| \leq j} \sum_{k=0}^{j-|\alpha|} i^{|\alpha|} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \sigma_{p-j+|\alpha|+k}^P \partial_x^\alpha \sigma_{-p-k}^{P^{-1}} \right) \quad (4.26)$$

We prove by induction that for any $j \in \mathbb{N}$, $\sigma_{-p-j}^{P^{-1}} \in E_{-p-j}$: suppose that for a $j \in \mathbb{N}^*$, we have for any $j' < j$, $\sigma_{-p-j'}^{P^{-1}} \in E_{-p-j'}$. We then directly check with Lemma 4.3.5 and (4.26) that $\sigma_{-p-j}^{P^{-1}} \in E_{-p-j}$.

The case $P \in \mathcal{B}_o$ is similar. \square

Lemma 4.3.7. *For any $k \in \mathbb{Z}$, $\mathcal{D}^k \in \mathcal{B}_e$ and when k is odd, $|\mathcal{D}|^k \in \mathcal{B}_o$.*

Proof. Since $\mathcal{D} \in \mathcal{B}_e$, \mathcal{D}^{-2} is in \mathcal{B}_e by Lemma 4.3.6 and 4.3.5 and so is \mathcal{D}^k .

Using (4.25) for the equation $|\mathcal{D}||\mathcal{D}| = \mathcal{D}^2$, we check that $\sigma_1^{|\mathcal{D}|}(x, \xi) = \|\xi\|_x$ and for any $j \in \mathbb{N}^*$,

$$\sigma_{1-j}^{|\mathcal{D}|} = \frac{1}{2\|\xi\|_x} \left(\sigma_{2-j}^{\mathcal{D}^2} - \sum_{0 < k < j} \sigma_{1-j+k}^{|\mathcal{D}|} \sigma_{1-k}^{|\mathcal{D}|} + \sum_{0 < |\alpha| \leq j} \sum_{k=0}^{j-|\alpha|} i^{|\alpha|} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \sigma_{1-j+|\alpha|+k}^{|\mathcal{D}|} \partial_x^\alpha \sigma_{1-k}^{|\mathcal{D}|} \right). \quad (4.27)$$

Again, a straightforward induction argument shows that for any $j \in \mathbb{N}$, $\sigma_{1-j}^{|\mathcal{D}|} \in O_{1-j}$, and thus $|\mathcal{D}| \in \mathcal{B}_o$. The result follows as above. \square

In the next four lemmas, we emphasize the fact that only some of the results could be obtained using the trick (4.11) with the operator J .

Lemma 4.3.8. (i) If d is odd, then for any $P \in \mathcal{B}_e$, $\oint P = 0$.

(ii) If d is even, then for any $P \in \mathcal{B}_o$, $\oint P = 0$.

(iii) For any pseudodifferential operator $P \in \Psi_1(\mathcal{A})$,

- when d is odd, then $\oint P = 0$,

- when d is even, then $\oint P|\mathcal{D}|^{-1} = 0$.

Proof. (i) Since $\sigma_{-d}^P \in E_{-d}$, $\sigma_{-d}^P(x, \xi) = \sum_{i \in I} \frac{\xi^{\beta^i}}{\|\xi\|_x^{2k_i}} h_i(x)$ where $|\beta^i|$ are odd. The integration on the cosphere in (4.12) therefore vanishes.

(ii) The same argument can be applied.

(iii) Direct consequence of (i) and (ii). \square

Remark 4.3.9. Lemma 4.3.8 (iii) entails for instance that $\oint B|\mathcal{D}|^{-(2k+1)}$ where B is a polynomial in \mathcal{A} and \mathcal{D} and $k \in \mathbb{N}$, always vanish in even dimension, while $\oint B\mathcal{D}^{-2k}$ always vanish in odd dimension. In other words, $\oint B|\mathcal{D}|^{-(d-q)} = 0$ for any odd integer q .

We shall now pay attention to the real or purely imaginary nature (independently of the appearance of gamma matrices) of homogeneous symbols of a given pseudodifferential operator. Let

$$\mathcal{C} := \{ P \in \Psi^p(M) : \sigma_{p-j}^P \in I_j, \forall j \in \mathbb{N} \}$$

where $I_k = I_e$ if k is even and $I_k = I_o$ if k is odd, with

$$I_e := \{ f \in C^\infty(U \times \mathbb{R}^n, \mathcal{M}_m(\mathbb{C})) : f = \gamma_{k_1} \cdots \gamma_{k_q} h(x, \xi), h \text{ real valued} \},$$

$$I_o := \{ f \in C^\infty(U \times \mathbb{R}^n, \mathcal{M}_m(\mathbb{C})) : f = i \gamma_{k_1} \cdots \gamma_{k_q} h(x, \xi), h \text{ real valued} \}.$$

Lemma 4.3.10. (i) \mathcal{C} is a sub-algebra of $\Psi(M)$.

(ii) If $P \in \mathcal{C}$ is hypo-elliptic then $P^{-1} \in \mathcal{C}$.

(iii) $\mathcal{D}^k \in \mathcal{C}$ and $|\mathcal{D}|^k \in \mathcal{C}$ for any $k \in \mathbb{Z}$.

Proof. (i) Consequence of (4.25).

(ii) Consequence of (4.26).

(iii) It is clear that $\mathcal{D} \in \mathcal{C}$ and the fact that $|\mathcal{D}| \in \mathcal{C}$ is a consequence of (4.27). \square

Lemma 4.3.11. Let $k \in \mathbb{N}$ odd. Then any element B of the polynomial algebra generated by \mathcal{A} and $[D, \mathcal{A}]$ satisfies $\oint B|\mathcal{D}|^{-(d-k)} = \oint BF|\mathcal{D}|^{-(d-k)} = 0$.

Proof. We may assume that B is selfadjoint so $\oint B\mathcal{D}^{-(d-k)} \in \mathbb{R}$.

By Lemma 4.3.10, $\sigma_{-d}^{B\mathcal{D}^{-(d-k)}} = \sigma_0^B \sigma_{-d}^{\mathcal{D}^{-(d-k)}} \in I_k$. Thus $\oint A\mathcal{D}^{-k} \in i\mathbb{R}$ and the result follows. The case $\oint BF|\mathcal{D}|^{-(d-k)}$ is similar. \square

We now look at the information given by the gamma matrices.

Lemma 4.3.12. *For any one-form A , $\oint A|\mathcal{D}|^{-q} = 0$, $q \in \mathbb{N}$ in either of the following cases:*

- $d \not\equiv 1 \pmod{8}$ and $d \not\equiv 5 \pmod{8}$,
- $(d \equiv 1 \pmod{8} \text{ or } d \equiv 5 \pmod{8})$ and $(q \text{ is even or } q \geq \frac{d+3}{2})$.

Proof. In the case $d \not\equiv 1 \pmod{8}$ and $d \not\equiv 5 \pmod{8}$, the result follows from the fact that $\varepsilon = 1$.

The case d even and q odd or d odd and q even is done by Lemma 4.3.8 (iii).

Suppose that d is even and q is even. If $q = 2k$, with a recurrence and the symbol product formula, we see that $\sigma_{2k-j}^{\mathcal{D}^{2k}}$ and all its derivatives are linear combinations of terms of the form $f(x, \xi) \otimes \gamma^{j_1} \dots \gamma^{j_i}$ where i is even and less than $2j$ (with the convention $\gamma^{j_1} \dots \gamma^{j_i} = 1$ if $i = 0$). We call (P_j) this property. The parametrix equation $\mathcal{D}^{2k}\mathcal{D}^{-2k} = 1$ entails that $\sigma_{-2k}^{\mathcal{D}^{-2k}} = (\sigma_{2k}^{\mathcal{D}^{2k}})^{-1}$ and for any $j \geq 1$,

$$\begin{aligned} \sigma_{-2k-j}^{\mathcal{D}^{-2k}} = & -\sigma_{-2k}^{\mathcal{D}^{-2k}} \left(\sum_{r=\max\{j-2k, 0\}}^{j-1} \sigma_{2k-(j-r)}^{\mathcal{D}^{2k}} \sigma_{-2k-r}^{\mathcal{D}^{-2k}} \right. \\ & \left. + \sum_{1 \leq |\alpha| \leq 2k} \sum_{r=\max\{j-2k, 0\}}^{j-|\alpha|} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_{\xi}^{\alpha} \sigma_{2k-(j-|\alpha|-r)}^{\mathcal{D}^{2k}} \partial_x^{\alpha} \sigma_{-2k-r}^{\mathcal{D}^{-2k}} \right). \end{aligned}$$

Note that $\sigma_{-2k}^{\mathcal{D}^{-2k}}$ satisfies (P_0) . By recurrence, this formula shows that $\sigma_{-2k-j}^{\mathcal{D}^{-2k}}$ satisfies (P_j) for any $j \in \mathbb{N}$. In particular, $\sigma_{-d}^{\mathcal{D}^{-2k}}$ satisfies (P_{-2k+d}) and the result follows then from (4.23) and the product of an odd number (different from the dimension) of gamma matrices is traceless.

Suppose now that d is odd, q is odd and $d \geq q$. In that situation, any odd number of gamma matrices $\gamma^{i_1} \dots \gamma^{i_r}$ is traceless when $r < d$.

Using (4.25) for the equation $|\mathcal{D}|^{-q}|\mathcal{D}|^{-q} = \mathcal{D}^{-2q}$, we check that $\sigma_{-q}^{|\mathcal{D}|^{-q}}(x, \xi) = \|\xi\|_x^{-q}$ and for any $j \in \mathbb{N}^*$,

$$\sigma_{-q-j}^{|\mathcal{D}|^{-q}} = \frac{1}{2\|\xi\|_x^{-q}} \left(\sigma_{-2q-j}^{\mathcal{D}^{-2q}} - \sum_{0 < k < j} \sigma_{-q-j+k}^{|\mathcal{D}|^{-q}} \sigma_{-q-k}^{|\mathcal{D}|^{-q}} + \sum_{0 < |\alpha| \leq j} \sum_{k=0}^{j-|\alpha|} i^{|\alpha|} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_{\xi}^{\alpha} \sigma_{-q-j+|\alpha|+k}^{|\mathcal{D}|^{-q}} \partial_x^{\alpha} \sigma_{-q-k}^{|\mathcal{D}|^{-q}} \right).$$

We saw that each $\sigma_{-2q-j}^{\mathcal{D}^{-2q}}$ satisfies (P_j) , that is to say, is a linear combination of terms of the form $f(x, \xi) \otimes \gamma^{j_1} \dots \gamma^{j_i}$ where i is even and less than $2j$. Again, a straightforward induction argument shows that for any $j \in \mathbb{N}$, $\sigma_{-q-j}^{|\mathcal{D}|^{-q}}$ satisfies (P_j) . In particular $\sigma_{-d}(A|\mathcal{D}|^{-q})$ is a linear combination of terms of the form $f(x, \xi) \otimes \gamma^{j_1} \dots \gamma^{j_r}$ where $r \leq 2(d-q) + 1$ is odd. This yields the result. \square

The fact that $\oint A\mathcal{D}^{-d+1} = 0$, consequence of Lemmas 4.3.8 and 4.3.11 is also a consequence of the fact that $\sigma_{-d}^{\mathcal{D}^{-d+1}}(x_0, \xi) = 0$:

Lemma 4.3.13. *For all $k \in \mathbb{N}^*$, we have $\sigma_{k-1}^{\mathcal{D}^k}(x_0, \xi) = \sigma_{-k-1}^{\mathcal{D}^{-k}}(x_0, \xi) = 0$.*

Proof. We already know that $\sigma_0^{\mathcal{D}}(x_0, \xi) = 0$, see (4.19). We proceed by recurrence, assuming $\sigma_{k-1}^{\mathcal{D}^k}(x_0, \xi) = 0$ for $k = 1, \dots, n$. Then $\sigma_n^{\mathcal{D}^{n+1}} = \sigma_n^{\mathcal{D}^n} \sigma_0^{\mathcal{D}} + \sigma_{n-1}^{\mathcal{D}^n} \sigma_1^{\mathcal{D}} - i \partial_{\xi_k} \sigma_n^{\mathcal{D}^n} \partial_{x_k} \sigma_1^{\mathcal{D}}$, thus by (4.19) and (4.20), $\sigma_n^{\mathcal{D}^{n+1}}(x_0, \xi) = 0$.

Since $\mathcal{D}\mathcal{D}^{-1} = 1$ yields $\sigma_{-2}^{\mathcal{D}^{-1}}(x_0, \xi) = -(\sigma_{-1}^{\mathcal{D}^{-1}} \sigma_0^{\mathcal{D}})(x_0, \xi) = 0$, we assume $\sigma_{-k-1}^{\mathcal{D}^{-k}}(x_0, \xi) = 0$ for $k = 1, \dots, n$. Then $\sigma_{-n-2}^{\mathcal{D}^{-n-1}} = \sigma_{-n}^{\mathcal{D}^{-n}} \sigma_{-2}^{\mathcal{D}^{-1}} + \sigma_{-n-1}^{\mathcal{D}^{-n}} \sigma_{-1}^{\mathcal{D}^{-1}} - i \partial_{\xi_k} \sigma_{-n}^{\mathcal{D}^{-n}} \partial_{x_k} \sigma_{-1}^{\mathcal{D}^{-1}}$. Using (4.22) and recurrence hypothesis, $\sigma_{-n-2}^{\mathcal{D}^{-n-1}}(x_0, \xi) = 0$. \square

Remark 4.3.14. *Regularity of $\zeta_X(s) := \text{Tr}(|X|^{-s})$ at point 0 when X is an elliptic selfadjoint differential operator of order one (see [64]):*

One checks that $\zeta_X(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}(e^{-t|X|}) dt$ for $\Re(s) > d$. Because of the asymptotic expansion

$$\text{Tr}(e^{-t|X|}) = t^{-d} \sum_{n=0}^N t^n a_n[X] + \mathcal{O}(t^{N+1-d}) \quad (4.28)$$

and meromorphic extension to the whole complex plane, $\text{Res}_{s=d-n} \zeta_X(s) = \frac{a_n[X]}{\Gamma(d-n)}$. In particular, $\zeta_X(s) = \Gamma(s)^{-1} \left(\frac{a_d[X]}{s} + f(s) \right)$, where f is holomorphic around $s = 0$. By (4.16) we get that $\zeta_X(s)$ is regular around zero and $\zeta_X(0) = a_d[X]$ if d is even and $\zeta_X(0) = 0$ if d is odd.

Corollary 4.3.15. $\zeta_{\mathcal{D}+A}(0) = \zeta_{\mathcal{D}}(0) = 0$ when $d = \dim(M)$ is odd.

When d is even, $\zeta_{\mathcal{D}+A}(0) - \zeta_{\mathcal{D}}(0) = \sum_{k=1}^{d/2} \frac{1}{2k} f(A \mathcal{D}^{-1})^{2k}$.

Proof. The result follows from (1.11) and Lemma 4.3.4. \square

A proof of (1.11) also follows from $\sigma^{\log(1+A\mathcal{D}^{-1})} \sim \sum_{k=1}^\infty \frac{(-1)^k}{k} \sigma^{(A\mathcal{D}^{-1})^k}$ with $\log(X) := \frac{\partial}{\partial z}|_{z=0} X^z$, so $\text{Wres}(\log(1+A\mathcal{D}^{-1})) = \sum_{k=1}^d \frac{(-1)^k}{k} \text{Wres}((A\mathcal{D}^{-1})^k)$ since $(A\mathcal{D}^{-1})^k$ has zero Wodzicki residue if $k > d$ and moreover $\zeta_{\mathcal{D}+A}(0) = -\text{Wres}(\log(\mathcal{D}+A))$. Actually, the important point is that $\det(X) := e^{\text{Wres}(\log(X))}$ is multiplicative (see [100]). Moreover, such determinant is different from the ζ -determinant $e^{-\zeta'_X(0)}$ used for instance by Hawking [77] in his regularization via the partition function which suffers from conformal anomalies.

The fact that in the asymptotic expansion of the heat kernel (4.28), the term $a_2[\mathcal{D}+A]$ depends only on the scalar curvature, so independent of A is reflected in

Lemma 4.3.16. *In any spectral triple of dimension 2 (commutative or not) with vanishing tadpoles of order zero (i.e. (1.19) is satisfied), $\zeta_{\mathcal{D}+A}(0) = \zeta_{\mathcal{D}}(0)$ for any one-form A .*

Proof. Let $a_1, a_2, b_1, b_2 \in \mathcal{A}$. Then, with $A_1 = a_1[\mathcal{D}, b_1]$,

$$\oint A_1 \mathcal{D}^{-1} a_2[\mathcal{D}, b_2] \mathcal{D}^{-1} = \oint A_1[\mathcal{D}^{-1}, a_2][\mathcal{D}, b_2] \mathcal{D}^{-1} + \oint A_1 a_2 \mathcal{D}^{-1}[\mathcal{D}, b_2] \mathcal{D}^{-1}.$$

The first term is zero since the integrand is in OP^{-3} , while the second term is equal to $\oint (a_1\alpha(b_1a_2) - a_1b_1\alpha(a_2))(\alpha(b_2) - b_2)$, so is zero using $\alpha(x)\alpha(y) = \alpha(xy)$, $\oint xy = \oint x\alpha(y)$ by (1.19) and the fact that \oint is a trace. Thus $\oint (A\mathcal{D}^{-1})^2 = 0$ and Corollary 4.3.15 yields the result. \square

Note that $\zeta_{\mathcal{D}+A}(0) - \zeta_{\mathcal{D}}(0)$ is usually non zero: consider for instance the flat 4-torus and as a generic selfadjoint one-form A , take

$$A := \phi \in [0, 2\pi]^4 \mapsto -i\gamma^\alpha \sum_{l \in \mathbb{Z}^4} a_{\alpha,l} e^{il^k \phi_k},$$

where $a_{\alpha,l}$ is in the Schwartz space $\mathcal{S}(Z^4)$ and $a_{\alpha,l} = -\overline{a_{\alpha,-l}}$. We have by Lemma 2.3.12, (with $c = \frac{8\pi^2}{3}$, $|l|^2 = \sum_k l^{k^2}$ and $\Theta = 0$)

$$\zeta_{\mathcal{D}+A}(0) - \zeta_{\mathcal{D}}(0) = \oint (A\mathcal{D}^{-1})^2 = c \sum_{l \in \mathbb{Z}^4} a_{\alpha_1,l} a_{\alpha_2,-l} (l^{\alpha_1} l^{\alpha_2} - \delta^{\alpha_1 \alpha_2} |l|^2)$$

since $\oint (A\mathcal{D}^{-1})^4 = 0$.

This last equality suggests that Lemma 4.3.16 can be extended:

Proposition 4.3.17. *For any one-form A , $\oint (A\mathcal{D}^{-1})^d = 0$ if $d = \dim(M)$.*

Proof. As in the proof of Lemma 4.3.16, \mathcal{D}^{-1} commutes with the element in the algebra as the integrand is in OP^{-d} . So for a family of $a_i, b_i \in \mathcal{A}$ and using $a := \prod_{i=1}^d a_i$,

$$\oint \prod_{i=1}^d (a_i [\mathcal{D}, b_i] \mathcal{D}^{-1}) = \oint \left(\prod_{i=1}^d a_i \right) \prod_{i=1}^d ([\mathcal{D}, b_i] \mathcal{D}^{-1}) = \oint a \prod_{i=1}^d (\alpha(b_i) - b_i).$$

We obtain, since $\alpha(b_i) - b_i \in OP^{-1}$,

$$\sigma_{-d}^{a \prod_{i=1}^d \alpha(b_i) - b_i} = a \prod_{i=1}^d \sigma_{-1}^{\alpha(b_i) - b_i} = a \prod_{i=1}^d \sigma_{-1}^{\alpha(b_i)}.$$

Moreover, $\sigma_{-1}^{\mathcal{D}b_i \mathcal{D}^{-1}}(x_0, \xi) = 0$: we already know by Lemma 4.3.13 that $\sigma_{-2}^{\mathcal{D}^{-1}}(x_0, \xi) = 0$, by (4.21) that $\partial_{x^k} \sigma_{-1}^{\mathcal{D}^{-1}}(x_0, \xi) = 0$ for all k , and $\sigma_0^{\mathcal{D}b_i}(x_0, \xi) = b_i(x_0) \sigma_0^{\mathcal{D}}(x_0, \xi) = 0$ giving the claim and the result. \square

This proposition does not survive in noncommutative spectral triples, see for instance [82, Table 1].

Note that for a one-form A , $\oint A^d \mathcal{D}^{-d} \neq \oint (A\mathcal{D}^{-1})^{-d} = 0$: in dimension $d = 2$, as in (4.24),

$$\oint A^2 \mathcal{D}^{-2} = -2c_d \operatorname{Tr}(\gamma^k \gamma^l) \int_M a_k a_l \nu_g.$$

It is known (see [37, Proposition 1.153]) that the $d-2$ term (for $d = 4$) in the spectral action expansion $\oint |\mathcal{D} + A|^{-2}$ is independent of the perturbation A . This is why the Einstein–Hilbert action $S(\mathcal{D}) = \oint |\mathcal{D}|^{-d+2} = -c \int_M \tau \sqrt{g} dx$ (see [68, Theorem 11.2]) is so fundamental. Here τ is the scalar curvature (positive on the sphere) and c is a positive constant.

We give here another proof of this result.

Lemma 4.3.18. *We have $\oint |\mathcal{D} + A|^{-d+2} = \oint |\mathcal{D}|^{-d+2} = -c \int_M \tau \sqrt{g} dx$ with $c = \frac{d-2}{24} \oint |\mathcal{D}|^{-d}$.*

Proof. We get from Lemma 1.3.10 (ii), the following equality, where $X := A\mathcal{D} + \mathcal{D}A + A^2$:

$$\oint |\mathcal{D} + A|^{-d+2} - \oint |\mathcal{D}|^{-d+2} = \frac{(d-2)}{2} \left(\frac{d}{4} \oint X^2 |\mathcal{D}|^{-d-2} - \oint X |\mathcal{D}|^{-d} \right).$$

Since the tadpole terms vanish, we have $\oint X|\mathcal{D}|^{-d} = \oint A^2|\mathcal{D}|^{-d}$. Moreover, since $\text{mod } OP^1$, $X^2 = (A\mathcal{D})^2 + (\mathcal{D}A)^2 + A\mathcal{D}^2A + \mathcal{D}A^2\mathcal{D}$, we get with $[\mathcal{D}^2, A] \in OP^1$,

$$\oint X^2|\mathcal{D}|^{-d-2} = 2\oint (A\mathcal{D})^2|\mathcal{D}|^{-d-2} + 2\oint A^2|\mathcal{D}|^{-d}$$

which yields

$$\oint |\mathcal{D} + A|^{-d+2} - \oint |\mathcal{D}|^{-d+2} = \frac{d(d-2)}{4} \left(\oint (A\mathcal{D})^2|\mathcal{D}|^{-d-2} - \frac{2-d}{d} \oint A^2|\mathcal{D}|^{-d} \right).$$

Thus, it is sufficient to check that

$$\int_{S_{x_0}^*M} \text{Tr}(\sigma_{-d}((A\mathcal{D})^2|\mathcal{D}|^{-d-2})(x_0, \xi)) d\xi = \frac{2-d}{d} \int_{S_{x_0}^*M} \text{Tr}(\sigma_{-d}(A^2|\mathcal{D}|^{-d})(x_0, \xi)) d\xi.$$

A straightforward computation yields, with $A =: -ia_\mu\gamma^\mu$, and $\sigma_1^{\mathcal{D}}(x_0, \xi) = \gamma^\mu\xi_\mu$,

$$\begin{aligned} \int_{S_{x_0}^*M} \sigma_{-d}((A\mathcal{D})^2|\mathcal{D}|^{-d-2})(x_0, \xi) d\xi &= -\frac{1}{d} a_\mu a_\tau \text{Tr}(\gamma^\mu\gamma^\nu\gamma^\tau\gamma_\nu) \text{Vol}(S^{d-1}), \\ \int_{S_{x_0}^*M} \sigma_{-d}(A^2|\mathcal{D}|^{-d})(x_0, \xi) d\xi &= -a_\mu a_\tau \text{Tr}(\gamma^\mu\gamma^\tau) \text{Vol}(S^{d-1}). \end{aligned}$$

Now, $\oint |\mathcal{D} + A|^{-d+2} = \oint |\mathcal{D}|^{-d+2}$ follows from the equality $\text{Tr}(\gamma^\mu\gamma^\nu\gamma^\tau\gamma_\nu) = (2-d) \text{Tr}(\gamma^\mu\gamma^\tau)$. The constant c is given in [68, Theorem 11.2 and normalization (11.2)]. \square

Remark 4.3.19. In [37, Definition 1.143], the above result justifies the definition of a scalar curvature for $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ as $\mathcal{R}(a) := \oint a|\mathcal{D}|^{-d+2}$ for $a \in \mathcal{A}$. This map is of course a trace on \mathcal{A} for a commutative geometry. But for the triple associated to $SU_q(2)$, this not a trace since

$$\mathcal{R}(aa^*) = \oint aa^*|\mathcal{D}|^{-1} = \frac{-q^4+6q^2+3}{2(1-q^2)^2} \quad \text{while} \quad \mathcal{R}(a^*a) = \oint a^*a|\mathcal{D}|^{-1} = \frac{3q^4+6q^2-1}{2(1-q^2)^2}.$$

Chapitre 5

Global pseudodifferential calculus on manifolds with linearization

5.1 Introduction

It has been proven by Gayral et al. in [59] that Moyal planes are (noncompact) spectral triples. In other words, one can consider Moyal planes as noncommutative Riemannian spin noncompact manifolds. A bridge between the world of deformation quantization and the world of noncommutative geometry has thus been constructed. This suggests that the paradigm of spectral triples is a good environment to deal with quantization problems. The Moyal product is defined on the Schwartz space $\mathcal{S}(\mathbb{R}^{2n})$ of rapidly decaying functions by

$$f \star g(x) := (\pi\theta)^{-2n} \int_{\mathbb{R}^{4n}} dy dz f(y) g(z) e^{\frac{2i}{\theta}(x-y) \cdot S(x-z)} \quad (5.1)$$

where $\theta \in \mathbb{R}^*$ and $S = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$, and gives to $\mathcal{S}(\mathbb{R}^{2n})$ a Fréchet pre- C^* -algebra structure. The noncompact spectral triple described in [59] is based on this algebra, and extensions to isospectral deformations have been established [61, 154].

The extension of this remarkable construction to more general symplectic manifolds, for instance when \mathbb{R}^{2n} is replaced by the cotangent bundle T^*M of a general manifold M is an open problem. We propose here the study of a pseudodifferential calculus that allows to extend the construction of the Moyal product to more general spaces. The main idea is to use a global pseudodifferential calculus on a manifold M that gives us a full algebra isomorphism between symbols and operators.

Classically, a pseudodifferential operator on a (smooth, finite dimensional) manifold is defined through local charts and the notion of pseudodifferential operator on open subsets of \mathbb{R}^n [127, 139]. In this setting, the full symbol of a pseudodifferential operator is a coordinate dependent notion. However, the *principal* symbol can be globally defined as a function on the cotangent bundle. Naturally, the question of a full coordinate free definition of the symbol calculus of pseudodifferential operators on a manifold has been considered. One approach, based on the ideas of Bokobza-Haggiag [9], Widom [148, 149] and Drager [49] allows such a calculus if one provides the manifold with a linear connection. Parallel transport along geodesics and the exponential map to connect any two points sufficiently close on the manifold are then used for the definitions and properties of local phase functions and oscillatory integrals. Safarov [120] has formulated a version

of a full coordinate free symbol calculus and λ -quantization ($0 \leq \lambda \leq 1$) using invariant oscillatory integral over the cotangent bundle and determined by the linear connection. Pflaum [111, 112] developed a complete symbol calculus on any Riemannian manifold using normal coordinates and microlocal lift on the test functions on manifolds with arbitrary Hermitian bundles. Sharafutdinov [125, 126] constructed a similar global pseudodifferential calculus, based on coordinate invariant geometric symbols. Further results in the same direction, connection to Weyl quantization and application to physics has been considered in Fulling and Kennedy [57], Fulling [56] and Güntürk [74]. Connection between complete symbol calculus, deformation quantization and star-products on the cotangent bundle has also been made (see for instance Gutt [75], Bordemann, Neumaier and Waldmann [10] and Voronov [145, 146]). Getzler [63] used a global pseudodifferential calculus in the context of the Atiyah-Singer index theorem on supermanifolds.

All these pseudodifferential calculi are based on symbol (functions of $(x, \theta) \in T^*M$) estimates over the covariable θ while the dependence on the variable x is only controlled locally uniformly on compact sets. This is well suited for the case of a compact manifold. For non-compact manifolds, we have to impose a uniform control over x in order to obtain $L^2(M)$ continuity of operators of order 0 and compactness of the remainder operators if the control over x is decaying. In other words, any global pseudodifferential calculus adapted to non-compact manifolds and sensitive to non-local effects needs to encode the behaviour “at infinity” of symbols. On the Euclidean space \mathbb{R}^n , several types of pseudodifferential calculi have been defined: standard pseudodifferential calculus with uniform control over x (see for instance Hörmander [81], Beals [5], Shubin [129]), isotropic calculus with simultaneous decay of the x and θ variables (Shubin [127, 128], Melrose [105]), and *SG*-pseudodifferential calculus with separated decay of the x and θ variables (Shubin [128], Parenti [109], Cordes [40, 41], Schrohe [122]), which is invariant under a special class of diffeomorphisms and can be extended to an adapted class of manifolds, namely the *SG*-manifolds (Schrohe [122]). This class of manifolds contains the non-compact manifolds “with exits” and adapted pseudodifferential calculus has been developed (see for instance Cordes [40], Schulze [132], Maniccia and Panarese [102]). Another approach, based on Lie structures at infinity, has been investigated to study the geometry of pseudodifferential operators on non-compact manifolds. Describing the geometry at infinity of the basis manifold by a Lie algebra of vector fields, an adapted pseudodifferential calculus has been constructed (see for instance Melrose [106], Mazzeo and Melrose [104], Ammann, Lauter and Nistor [3]). Let us also mention the groupoid approach: by associating to any manifold with corners a smooth Lie groupoid and by building a pseudodifferential calculus on Lie groupoids, the *b*-calculus of Melrose on manifolds with corners can be generalized (see Monthubert [107]).

Our purpose is to construct a global pseudodifferential calculus that generalizes the standard and *SG* calculi on \mathbb{R}^n , on manifolds with linearization. These manifolds provide a natural geometric setting to deal simultaneously with the questions of a global isomorphism between symbols and pseudodifferential operators, and the non-local effects associated to non-compact manifolds.

We define in section 2 a manifold with linearization (or exponential manifold) as a pair (M, \exp) where M is a smooth real finite-dimensional manifold and \exp is an abstract exponential map, a smooth map from the tangent bundle onto M that satisfies, besides the usual properties of an exponential map associated to a connection ∇ on TM , the property that at each point $x \in M$, \exp_x is a diffeomorphism. Any Cartan–Hadamard manifold with its canonical exponential map is an exponential manifold. These diffeomorphisms are used to define topological vector spaces of functions on the manifold (or on TM , T^*M , $M \times M$) that generalize, for instance, the notions of

rapidly decaying function on \mathbb{R}^n or of tempered distribution, provided that we add a hypothesis of “ \mathcal{O}_M -bounded geometry” on the exponential map. In section 3, we use linearizations in the spirit of Bokobza-Haggiag [9], to define symbol and quantization maps. This leads to topological isomorphisms between tempered distributional sections on T^*M and $M \times M$, if we consider polynomially controlled (at infinity) linearizations (\mathcal{O}_M -linearizations). In particular, we extend the usual (explicit) Moyal product (or λ -product, for the λ -quantization) on any exponential manifold with \mathcal{O}_M -bounded geometry on which we set a \mathcal{O}_M -linearization. We get the following λ -product formula, giving a Fréchet algebra structure to $\mathcal{S}(T^*M)$,

$$a \circ_\lambda b(x, \eta) = \int_{T_x(M) \times M} d\mu_x(\xi) d\mu(y) \int_{V_{x, \xi, y}^\lambda} d\mu_{x, \xi, y}^*(\theta, \theta') g_{x, \xi, y}^\lambda e^{2\pi i \omega_{x, \xi, y}^\lambda(\eta, \theta, \theta')} a(y_{x, \xi}^\lambda, \theta) b(y_{x, -\xi}^{1-\lambda}, \theta')$$

where $a, b \in \mathcal{S}(T^*M)$ and the other notations are detailed in Proposition 5.3.11.

In section 4, we define the symbol and amplitudes spaces for our pseudodifferential calculus. Symbol spaces can be defined in an intrinsic way on the exponential manifold with the help of “symbol-like” control (S_σ -bounded geometry, see Definition 5.2.8) of the coordinate change diffeomorphisms $\psi_{z, z'}^{\mathbf{b}, \mathbf{b}'}$ associated to the exponential map \exp on M . For practical reasons the definition of amplitudes here is slightly different from the usual functions of the parameters x, y and θ . Instead, our amplitudes generalize functions of the form $(x, \zeta, \vartheta) \mapsto a(x, x + \zeta, \vartheta)$, where a is a standard amplitude of the Euclidian pseudodifferential calculus. We establish continuity and regularity results for operators of the following form (which can be seen, for some forms of Γ , as special Fourier integral operators on \mathbb{R}^n):

$$\langle \mathfrak{Op}_\Gamma(a), u \rangle := \int_{\mathbb{R}^{3n}} e^{2\pi i \langle \vartheta, \zeta \rangle} \text{Tr} (a(x, \zeta, \vartheta) \Gamma(u)^*(x, \zeta)) d\zeta d\vartheta dx$$

where Γ is a topological isomorphism on $\mathcal{S}(\mathbb{R}^{2n}, L(E_z))$ (here E_z is a fixed fiber of the Hermitian bundle $E \rightarrow M$, so $L(E_z)$ can be identified with $\mathcal{M}_{\dim E_z}(\mathbb{C})$), a is in a $\mathcal{O}_{f, z}$ space (see Definition 5.4.13) and $u \in \mathcal{S}(\mathbb{R}^n, E_z)$. In particular, results of Proposition 5.4.14 and 5.4.17 and Lemma 5.4.18 are believed to be new.

With the help of a hypothesis of a control of symbol type over the derivative of the linearization (S_σ -linearizations), we obtain in section 5.4.4 an intrinsic definition (Theorem 5.4.30) of pseudodifferential operators $\Psi_\sigma^{l, m}$ on M . We see in section 5.4.5 a condition (H_V) on the linearization that entails that any pseudodifferential operator on M , when transferred in a frame (z, \mathbf{b}) , is a standard pseudodifferential operator on \mathbb{R}^n . This condition yields a L^2 -continuity result in Proposition 5.4.36. The last part of section 4 is devoted to the derivation of a symbol product asymptotic formula for the composition of two pseudodifferential operators. The main result is Theorem 5.4.47: under a special hypothesis (C_σ) on the linearization (see Definition 5.4.37), we have the following asymptotic formula for the normal symbol (transferred in a frame (z, \mathbf{b})) of the product of two pseudodifferential operators

$$\sigma_0(AB)_{z, \mathbf{b}} \sim \sum_{\beta, \gamma \in \mathbb{N}^n} c_\beta c_\gamma \partial_{\zeta, \vartheta}^{\gamma, \gamma} (a(x, \vartheta) \partial_{\zeta'}^\beta (e^{2\pi i \langle \vartheta, \varphi_{x, \zeta}(\zeta') \rangle} (\partial_{\vartheta'}^\beta f_b)(x, \zeta, \zeta', L_{x, \zeta}(\vartheta))))_{\zeta'=0} \tau_{x, \zeta}^{-1} \Big|_{\zeta=0}$$

where $a := \sigma_0(A)_{z, \mathbf{b}}$, $b := \sigma_0(B)_{z, \mathbf{b}}$, and other notations are defined in section 5.4.6.

Finally, we give in Section 5.5 two possible settings (besides the usual standard calculus on the Euclidian \mathbb{R}^n) in which the previous calculus applies. The first is based on the Euclidian

space \mathbb{R}^n , with a “deformed” (non-bilinear, non-flat) S_σ -linearization. The second example is the hyperbolic plane (or Poincaré half-plane) \mathbb{H} . We prove in particular that \mathbb{H} has a S_1 -bounded geometry. This allows to define a global Fourier transform, Schwartz spaces $\mathcal{S}(\mathbb{H})$, $\mathcal{S}(T^*\mathbb{H})$, $\mathcal{S}(T\mathbb{H})$, $\mathcal{B}(\mathbb{H})$ and the space of symbols $S_1^{l,m}(T^*\mathbb{H})$. Moreover we can then define in an intrinsic way a global complete pseudodifferential calculus on \mathbb{H} , and Moyal product, for any specified S_σ -linearization on \mathbb{H} .

5.2 Manifolds with linearization and basic function spaces

5.2.1 Abstract exponential maps, definitions and notations

The notion of linearization on a manifold was first introduced by Bokobza-Haggiag in [9] and corresponds to a smooth map ν from $M \times M$ into TM such that $\pi \circ \nu = \pi_1$, $\nu(x, x) = 0$ for any $x \in M$ and $(d_y \nu)_{y=x} = \text{Id}_{T_x M}$. In all the following, we shall work with “global” linearizations, in the following sense:

Definition 5.2.1. A manifold with linearization (or exponential manifold) is a pair (M, \exp) where M is a smooth manifold and \exp a smooth map from TM into M such that:

- (i) for any $x \in M$, $\exp_x : T_x M \rightarrow M$ defined as $\exp_x(\xi) := \exp(x, \xi)$, is a global diffeomorphism between $T_x M$ and M ,
- (ii) for any $x \in M$, $\exp_x(0) = x$ and $(d \exp_x)_0 = \text{Id}_{T_x M}$.

The map \exp will be called the exponential map, and $(x, y) \mapsto \exp_x^{-1}(y)$ the linearization, of the exponential manifold (M, \exp) . We shall sometimes use the shorthand $e_x^\xi := \exp_x(\xi)$.

Note that the term “exponential manifold” used here is not to be confused with the notion of “exponential statistical manifold” used in stochastic analysis. Remark that if $\exp \in C^\infty(TM, M)$ satisfies (i), then defining $\text{Exp} := \exp \circ T$ where $T(x, \xi) := \exp_x^{-1}(x) + (d \exp_x^{-1})_x \xi$, we see that (M, Exp) is an exponential manifold.

We will say that (M, ∇) (resp. (M, g)) is exponential, where M is a smooth manifold with connection ∇ on TM (resp. with pseudo-Riemannian metric g), if (M, \exp) where \exp is the canonical exponential map associated to ∇ (resp. to g) is an exponential manifold, or in other words, if for any $x \in M$, \exp_x is a diffeomorphism from $T_x M$ onto M . Note that (M, ∇) (resp. (M, g)) is exponential if and only if

- M is geodesically complete
- For any $x, y \in M$, there exists one and only one maximal geodesic γ such that $\gamma(0) = x$ and $\gamma(1) = y$.
- For any $x \in M$, \exp_x is a local diffeomorphism.

Remark 5.2.2. \mathbb{R}^n (with its standard metric of signature $(p, n - p)$) is an exponential manifold and any n -dimensional real exponential manifold is diffeomorphic to \mathbb{R}^n . In particular, an exponential manifold cannot be compact. A Cartan–Hadamard manifold is a Riemannian, complete, simply connected manifold with nonpositive sectional curvature. It is a consequence of the Cartan–Hadamard theorem (see for instance [97, Theorem 3.8]) that any Cartan–Hadamard manifold is exponential.

Remark 5.2.3. The exponential structure can be transported by diffeomorphism: if (M, \exp_M) is an exponential manifold, N a smooth manifold and $\varphi : M \rightarrow N$ is a diffeomorphism, then $(N, \exp_N := \varphi \circ \exp_M \circ T\varphi^{-1})$ is an exponential manifold.

Assumption 5.2.4. *We suppose from now on that (M, \exp) is an exponential n -dimensional real manifold.*

For any $x, y \in M$, we define γ_{xy} as the curve $\mathbb{R} \rightarrow M$, $t \mapsto \exp_x(t \exp_x^{-1} y)$, and $\tilde{\gamma}_{xy}(t) := \gamma_{yx}(1-t)$. Note that $\gamma_{xy}(0) = x$ and $\gamma_{xy}(1) = y$. If the exponential map is derived from a linear connection, we have for any $t \in \mathbb{R}$, $\gamma_{xy}(t) = \tilde{\gamma}_{xy}(t)$. In the general case, this is only true for $t = 0$ and $t = 1$.

The abstract exponential map \exp provides the manifold M with a notion of “points at infinity” and “straight lines” (γ_{xy}). It can be seen as a generalization to manifolds of the useful properties of \mathbb{R}^n for the study of the behaviour of functions at infinity. The abstract exponential map \exp formalizes the fact that our straight lines never stop and connect any two different points.

The diffeomorphism \exp_z^{-1} , for a given $z \in M$, is not stricto sensu a chart, since it maps M onto $T_z M$, which is diffeomorphic but not equal to \mathbb{R}^n . In order to obtain a chart, one needs to choose a linear basis of $T_z M$. If $z \in M$ and \mathbf{b} is a basis of $T_z M$ we will call the pair (z, \mathbf{b}) a (normal) frame. For any frame (z, \mathbf{b}) , we define $n_z^{\mathbf{b}} := L_{\mathbf{b}} \circ \exp_z^{-1}$ with $L_{\mathbf{b}}$ the linear isomorphism from $T_z M$ onto \mathbb{R}^n associated to \mathbf{b} . As a consequence, the pair $(M, n_z^{\mathbf{b}})$ is a chart which is a global diffeomorphism from M onto \mathbb{R}^n .

We denote $\psi_{z,z'}^{\mathbf{b},\mathbf{b}'} := n_z^{\mathbf{b}} \circ (n_{z'}^{\mathbf{b}'})^{-1}$ the normal coordinate change diffeomorphism from \mathbb{R}^n onto \mathbb{R}^n and $(\partial_{i,z,\mathbf{b}})_{i \in \mathbb{N}_n}$ and $(dx^{i,z,\mathbf{b}})_{i \in \mathbb{N}_n}$ (with $\mathbb{N}_n := \{1, \dots, n\}$) the global frame vector fields and 1-forms associated to the chart $n_z^{\mathbf{b}}$. We also note $n_{z,*}^{\mathbf{b}}$ the diffeomorphism from T^*M onto \mathbb{R}^{2n} defined by $n_{z,*}^{\mathbf{b}}(x, \theta) = (n_z^{\mathbf{b}}(x), \widetilde{M}_{z,x}^{\mathbf{b}}(\theta))$ where $(\widetilde{M}_{z,x}^{\mathbf{b}}(\theta)_i)_{i \in \mathbb{N}_n}$ are the components of θ in $(dx^{i,z,\mathbf{b}})_{i \in \mathbb{N}_n}$ and $n_{z,T}^{\mathbf{b}} : (x, \xi) \rightarrow (n_z^{\mathbf{b}}(x), M_{z,x}^{\mathbf{b}}(\xi))$ the diffeomorphism from TM onto \mathbb{R}^{2n} , where $(M_{z,x}^{\mathbf{b}}(\xi)_i)_{i \in \mathbb{N}_n}$ are the coordinates of ξ in the basis $(\partial_{i,z,\mathbf{b}_x})_{i \in \mathbb{N}_n}$. We have $M_{z,x}^{\mathbf{b}} = (dn_z^{\mathbf{b}})_x$ and $\widetilde{M}_{z,x}^{\mathbf{b}} = {}^t(dn_z^{\mathbf{b}})_x^{-1}$. The diffeomorphism from $M \times M$ onto \mathbb{R}^{2n} defined by $(x, y) \mapsto (n_z^{\mathbf{b}}(x), n_z^{\mathbf{b}}(y))$ will be noted $n_{z,M^2}^{\mathbf{b}}$.

We denote $(\partial_{i,z,\mathbf{b}})_{i \in \mathbb{N}_{2n}}$ the family of vector fields on T^*M (resp. TM , $M \times M$) associated to the chart $n_{z,*}^{\mathbf{b}}$ (resp. $n_{z,T}^{\mathbf{b}}$, $n_{z,M^2}^{\mathbf{b}}$) onto \mathbb{R}^{2n} . We suppose in all the following that \mathfrak{E} is an arbitrary normed finite dimensional complex vector space. If ν is a $(2n)$ -multi-index, we define the following operator on $C^\infty(T^*M, \mathfrak{E})$ (resp. $C^\infty(TM, \mathfrak{E})$, $C^\infty(M \times M, \mathfrak{E})$):

$$\partial_{z,\mathbf{b}}^\nu := \prod_{k=1}^{2n} \partial_{k,z,\mathbf{b}}^{\nu_k}.$$

If α and β are n -multi-indices, we denote (α, β) the $2n$ -multi-index obtained by concatenation. If α is a n -multi-index, $\partial_{z,\mathbf{b}}^\alpha$ is a linear operator on $C^\infty(M, \mathfrak{E})$. We fix the shortcut $\langle x \rangle := (1 + \|x\|^2)^{1/2}$ for any $x \in \mathbb{R}^p$, $p \in \mathbb{N}$. We will use the convention $x^\alpha := x_1^{\alpha_1} \cdots x_p^{\alpha_p}$ for $x \in \mathbb{R}^p$ and α p -multi-index, with $0^0 := 1$. If f is continuous function from \mathbb{R}^p to a normed vector space and g is a continuous function from \mathbb{R}^p to \mathbb{R} , we denote $f = \mathcal{O}(g)$ if and only if there exist $r > 0$, $C > 0$ such that for any $x \in \mathbb{R}^p \setminus B(0, r)$, $\|f(x)\| \leq C|g(x)|$. In the case where g is strictly positive on \mathbb{R}^p , this is equivalent to: there exists $C > 0$ such that for any $x \in \mathbb{R}^p$, $\|f(x)\| \leq Cg(x)$. We also introduce the following shorthands, for given (z, \mathbf{b}) , $x, y \in M$, $\theta \in T_x^*(M)$, $\xi \in T_x(M)$:

$$\begin{aligned} \langle x \rangle_{z,\mathbf{b}} &:= \langle n_z^{\mathbf{b}}(x) \rangle, & \langle \theta \rangle_{z,\mathbf{b},x} &:= \langle \widetilde{M}_{z,x}^{\mathbf{b}}(\theta) \rangle, & \langle \xi \rangle_{z,\mathbf{b},x} &:= \langle M_{z,x}^{\mathbf{b}}(\xi) \rangle, \\ \langle x, y \rangle_{z,\mathbf{b}} &:= \langle (n_z^{\mathbf{b}}(x), n_z^{\mathbf{b}}(y)) \rangle, & \langle x, \theta \rangle_{z,\mathbf{b}} &:= \langle (n_z^{\mathbf{b}}(x), \widetilde{M}_{z,x}^{\mathbf{b}}(\theta)) \rangle, & \langle x, \xi \rangle_{z,\mathbf{b}} &:= \langle (n_z^{\mathbf{b}}(x), M_{z,x}^{\mathbf{b}}(\xi)) \rangle. \end{aligned}$$

If f and g are in $C^0(\mathbb{R}^p, \mathbb{R}^{p'})$ we denote $f \asymp g$ the equivalence relation defined by: $\langle f \rangle = \mathcal{O}(\langle g \rangle)$ and $\langle g \rangle = \mathcal{O}(\langle f \rangle)$.

5.2.2 Parallel transport on an Hermitian bundle

Let E be an hermitian vector bundle (with typical fiber \mathbb{E} as a finite dimensional complex vector space) on the exponential manifold (M, \exp) . E admits a (non-unique) connection ∇^E compatible with the hermitian metric [7]. It is a differential operator from $C^\infty(M, E)$ (the space of smooth sections of $E \rightarrow M$) to $C^\infty(M, T^*M \otimes E)$ such that for any smooth function f on M and smooth E -sections ψ, ψ' ,

$$\begin{aligned}\nabla^E(f\psi) &= df \otimes \psi + f\nabla^E\psi, \\ d(\psi|\psi') &= (\nabla^E\psi|\psi') + (\psi|\nabla^E\psi'),\end{aligned}$$

where $(\psi|\psi')$ is the hermitian pairing of ψ and ψ' . We will note $|\psi|^2 := (\psi|\psi)$. The sesquilinear form $(\cdot|\cdot)_x$ of E_x is antilinear in the second variable by convention. The operator ∇^E can be (uniquely) extended as an operator acting on E -valued differential forms on M . If γ is a curve on M defined on an interval J and γ^*E the associated pullback bundle on J , there exists a natural connection (the pullback of ∇^E) on γ^*E , noted ∇^{γ^*E} compatible with ∇^E .

Let us fix $x, y \in M$ and $\gamma : J \rightarrow M$ a curve such that $\gamma(0) = x$ and $\gamma(1) = y$. For any $v \in E_x$, there exists a unique smooth section β of $\gamma^*E \rightarrow J$ such that $\beta(0) = v$ and $\nabla^{\gamma^*E}\beta = 0$. Clearly, $\beta(1) \in E_y$ and we can define a linear isomorphism τ_γ from E_x to E_y as $\tau_\gamma(v) = \beta(1)$. The map τ_γ is the parallel transport map associated to γ from E_x to E_y . The compatibility of ∇^E with the hermitian metric entails that the maps τ_γ are in fact isometries for the hermitian structures on E_x and E_y .

The vector bundle $L(E) \rightarrow M$, defined by $L(E)_x := L(E_x)$ (the space of endomorphisms on E_x), is lifted to T^*M , TM and $M \times M$ by setting the fiber at (x, θ) to $L(E_x)$ for T^*M or TM , and the fiber at (x, y) to $L(E_y, E_x)$ for $M \times M$. The canonical projection from T^*M or TM to M is noted π .

We denote $\tau_{xy} := \tau_{\gamma_{xy}}$. Remark that $\tau_{xy}^{-1} = \tau_{\tilde{\gamma}_{yx}}$. We define $\tau_z : x \mapsto \tau_{zx}$ and $\tau_z^{-1} : x \mapsto \tau_{zx}^{-1} = \tau_{zx}^*$.

If $u \in C^\infty(M, E)$ and $z \in M$, we denote $u^z(x) := (\tau_z^{-1}u)(x)$ for any $x \in M$. If a is a section of $L(E) \rightarrow T^*M$ or $L(E) \rightarrow TM$, we denote $a^z := (\tau_z^{-1} \circ \pi)a(\tau_z \circ \pi)$. If a is a section of $L(E) \rightarrow M \times M$, we denote $a^z(x, y) := \tau_z^{-1}(x)a(x, y)\tau_z(y)$. We also define $\tau^z := (x, y) \mapsto \tau_z^{-1}(y)\tau(x, y)\tau_z(x) \in L(E_z)$. Noting $\pi_1(x, y) := x$, $\pi_2(x, y) := y$, we get $a^z = (\tau_z^{-1} \circ \pi_1)a(\tau_z \circ \pi_2)$ and $\tau^z = (\tau_z^{-1} \circ \pi_2)a(\tau_z \circ \pi_1)$.

Parallel transport on E has the following smoothness property:

- Lemma 5.2.5.** (i) The map $\tau : (x, y) \mapsto \tau_{xy}$ (resp. $\tau^{-1} : (x, y) \mapsto \tau_{xy}^{-1}$) is a smooth section of the vector bundle $L(E)^\vee \rightarrow M \times M$ where the fiber at (x, y) is $L(E_x, E_y)$ (resp. of the vector bundle $L(E) \rightarrow M \times M$).
- (ii) $\tau_z \in C^\infty(M, L(E_z, E))$ and $\tau_z^{-1} \in C^\infty(M, L(E, E_z))$.
- (iii) $\tau^z \in C^\infty(M \times M, L(E_z))$.

Proof. (i) The map $G : TM \rightarrow M \times M$ defined by $G(v) := (\pi(v), \exp(v))$ is a local diffeomorphism since the Jacobian of G at $v_0 = (x_0, \xi_0) \in TM$ is equal to the Jacobian of \exp_{x_0}

at ξ_0 . Since it is also bijective (with inverse $G^{-1}(x, y) := (x, \exp_x^{-1}(y))$), it is a (global) diffeomorphism $TM \rightarrow M \times M$. The map $b(x, y, t) := (x, t \exp_x^{-1}(y))$ is thus a smooth map from $M \times M \times \mathbb{R}$ to TM , and we get a smooth parametrization by $M \times M$ of the following family of curves: $c(x, y) \mapsto (\gamma_{xy} : t \mapsto \exp b(x, y, t))$. This parametrization leads (see [50, p. 17]) to a smooth bundle homomorphism between $c^*(\cdot)(0)E \rightarrow M \times M$ and $c^*(\cdot)(1)E \rightarrow M \times M$, so a smooth section $\tau : (x, y) \mapsto \tau_{xy}$ of $L(E_x, E_y) \rightarrow M \times M$. The case of τ^{-1} is similar, by taking $b^{-1}(x, y, t) := b(x, y, 1 - t)$.

(ii, iii) are straightforward consequences of (i). \square

Corollary 5.2.6. *If u is in the space $C^\infty(M, E)$, then $u^z \in C^\infty(M, E_z)$. Similarly, if $a \in C^\infty(T^*M, L(E))$ (resp. $C^\infty(TM, L(E))$, $C^\infty(M \times M, L(E))$), then $a^z \in C^\infty(T^*M, L(E_z))$ (resp. $C^\infty(TM, L(E_z))$, $C^\infty(M \times M, L(E_z))$).*

Remark 5.2.7. *The vector bundle E on M is trivializable and the parallel transport provides a M -indexed family of trivializations, since for any $z \in M$, the pair $f_z : E \mapsto M \times \mathbb{E}, (x, v) \mapsto (x, \tau_{xz}(v))$, $\text{Id} : M \mapsto M, x \mapsto x$, is a vector bundle isomorphism from $E \rightarrow M$ onto $M \times \mathbb{E} \rightarrow M$. Note that if \exp is derived from a connection, $\tau_{xy}^{-1} = \tau_{yx}$ for any $x, y \in M$.*

5.2.3 \mathcal{O}_M and S_σ -bounded geometry

Classically, in Riemannian geometry, bounded geometry hypothesis gives boundedness on the covariant derivative of the Riemann curvature of the basis manifold. For the following pseudodifferential calculus, we shall need some hypothesis of that kind, formulated not with the curvature but with the exponential diffeomorphisms (“normal” coordinate transition maps). The hypothesis that we will need for pseudodifferential symbol calculus is actually not simply the boundedness condition on the derivatives of the transition maps, which is a classical consequence of bounded geometry. For symbol calculus, we will require that the n^{th} -derivatives are not only bounded, but decrease to zero at infinity as $\|x\|^{-\sigma(n-1)}$ where σ is a parameter in $[0, 1]$. Or, in other words, the normal coordinate change maps behave as “symbols” of order 1. Thus, we introduce the following

Definition 5.2.8. Let $\sigma \in [0, 1]$. The exponential manifold (M, \exp) is said to have a S_σ -bounded geometry if for any (z, \mathbf{b}) , (z', \mathbf{b}') , and any n -multi-index $\alpha \neq 0$,

$$(S_\sigma 1) \quad \partial^\alpha \psi_{z, z'}^{\mathbf{b}, \mathbf{b}'}(x) = \mathcal{O}(\langle x \rangle^{-\sigma(|\alpha|-1)}),$$

and a \mathcal{O}_M -bounded geometry if for any (z, \mathbf{b}) , (z', \mathbf{b}') , and any n -multi-index α , there exist $p_\alpha \geq 1$ such that

$$(\mathcal{O}_M 1) \quad \partial^\alpha \psi_{z, z'}^{\mathbf{b}, \mathbf{b}'}(x) = \mathcal{O}(\langle x \rangle^{p_\alpha}).$$

We shall be working with \mathcal{O}_M -bounded geometry for the definition of function spaces and Fourier transform and with S_σ -bounded geometry (for a $\sigma \in [0, 1]$) for pseudodifferential symbol calculus.

Definition 5.2.9. The triple (M, \exp, E) where (M, \exp) is exponential and E is a hermitian vector bundle on M has a S_σ -bounded geometry if (M, \exp) has a S_σ -bounded geometry and for any (z, \mathbf{b}) , z', z'' , and any n -multi-index α ,

$$(S_\sigma 2) \quad \partial_{z, \mathbf{b}}^\alpha \tau_{z'}^{-1} \tau_{z''}(x) = \mathcal{O}(\langle x \rangle_{z, \mathbf{b}}^{-\sigma|\alpha|}),$$

and a \mathcal{O}_M -bounded geometry if (M, \exp) has a \mathcal{O}_M -bounded geometry and for any (z, \mathbf{b}) , (z', \mathbf{b}') , and any n -multi-index α , there exist $p_\alpha \geq 1$ such that

$$(\mathcal{O}_M 2) \quad \partial_{z, \mathbf{b}}^\alpha \tau_{z'}^{-1} \tau_{z''}(x) = \mathcal{O}(\langle x \rangle_{z, \mathbf{b}}^{p_\alpha}).$$

Clearly, if $\sigma \leq \sigma'$, since $(S_{\sigma'} i) \Rightarrow (S_\sigma i)$, we have $S_{\sigma'}$ -bounded $\Rightarrow S_\sigma$ -bounded $\Rightarrow \mathcal{O}_M$ -bounded. Note that S_σ -bounded geometry on the vector bundle entails that the derivatives of the transport transition maps $\tau_z^{-1} \tau_{z'}$ (smooth from M to $L(E_{z'}, E_z)$) are bounded (for S_0 -bounded geometry) or decrease to zero with an order equal to the order of the derivative (for S_1 -bounded geometry). Remark also that if E is a trivial bundle and $\nabla^E = d$, then $(S_1 2)$ is automatically satisfied since the maps τ_z are all equal to the constant $x \mapsto Id_{\mathbb{E}}$.

Lemma 5.2.10. *Let $\sigma \in [0, 1]$ and (z, \mathbf{b}) , (z', \mathbf{b}') be given frames.*

(i) *If (M, \exp) has a S_σ -bounded geometry, there exist $K, C, C' > 0$ such that for any $x \in \mathbb{R}^n$, $x \in M$, $\theta \in T_x^*(M)$, $\xi \in T_x(M)$,*

$$\psi_{z, z'}^{\mathbf{b}, \mathbf{b}'} \asymp Id_{\mathbb{R}^n} \quad \text{and} \quad \langle x \rangle_{z, \mathbf{b}} \leq K \langle x \rangle_{z', \mathbf{b}'}, \quad (5.2)$$

$$\langle \theta \rangle_{z, \mathbf{b}, x} \leq C \langle \theta \rangle_{z', \mathbf{b}', x} \quad \text{and} \quad \langle \xi \rangle_{z, \mathbf{b}, x} \leq C' \langle \xi \rangle_{z', \mathbf{b}', x}, \quad (5.3)$$

and if (M, \exp) has a \mathcal{O}_M -bounded geometry, there exist $K, K', K'', C, C' > 0$ and $q \geq 1$ such that for any $x \in \mathbb{R}^n$, $x \in M$, $\theta \in T_x^(M)$, $\xi \in T_x(M)$,*

$$K' \langle x \rangle^{1/q} \leq \langle \psi_{z, z'}^{\mathbf{b}, \mathbf{b}'}(x) \rangle \leq K'' \langle x \rangle^q \quad \text{and} \quad \langle x \rangle_{z, \mathbf{b}} \leq K \langle x \rangle_{z', \mathbf{b}'}^q, \quad (5.4)$$

$$\langle \theta \rangle_{z, \mathbf{b}, x} \leq C \langle x \rangle_{z', \mathbf{b}'}^q \langle \theta \rangle_{z', \mathbf{b}', x} \quad \text{and} \quad \langle \xi \rangle_{z, \mathbf{b}, x} \leq C' \langle x \rangle_{z', \mathbf{b}'}^q \langle \xi \rangle_{z', \mathbf{b}', x}, \quad (5.5)$$

(ii) *For any given n -multi-indices α , we can write*

$$\partial_{z, \mathbf{b}}^\alpha = \sum_{0 \leq |\alpha'| \leq |\alpha|} f_{\alpha, \alpha'} \partial_{z', \mathbf{b}'}^{\alpha'}$$

where the $(f_{\alpha, \alpha'})$ are smooth real functions on M such that for each n -multi-indices α, α' ,

(a) *if (M, \exp) has a S_σ -bounded geometry, there exists $C_\alpha > 0$ such that for any $x \in M$, $|f_{\alpha, \alpha'}(x)| \leq C_\alpha \langle x \rangle_{z, \mathbf{b}}^{-\sigma(|\alpha| - |\alpha'|)}$,*

(b) *if (M, \exp) has a \mathcal{O}_M -bounded geometry, there exist $C_\alpha > 0$ and $q_\alpha \geq 1$ such that for any $x \in M$, $|f_{\alpha, \alpha'}(x)| \leq C_\alpha \langle x \rangle_{z, \mathbf{b}}^{q_\alpha}$.*

Proof. (i) Suppose that (M, \exp) has a S_σ -bounded geometry. Taylor formula implies that $\|\psi_{z, z'}^{\mathbf{b}, \mathbf{b}'}(x)\| \leq \|\psi_{z, z'}^{\mathbf{b}, \mathbf{b}'}(0)\| + C_0 \|x\|$ for any $x \in \mathbb{R}^n$, where $C_0 := \sup_{x \in \mathbb{R}^n} \|(d\psi_{z, z'}^{\mathbf{b}, \mathbf{b}'})_x\|$. As a consequence $\psi_{z, z'}^{\mathbf{b}, \mathbf{b}'}(x) = \mathcal{O}(\|x\|)$ and thus, there is $K'' > 0$ such that $\langle \psi_{z, z'}^{\mathbf{b}, \mathbf{b}'}(x) \rangle \leq K'' \langle x \rangle$. The same argument for $\psi_{z', z}^{\mathbf{b}', \mathbf{b}} = (\psi_{z, z'}^{\mathbf{b}, \mathbf{b}'})^{-1}$ gives $\psi_{z, z'}^{\mathbf{b}, \mathbf{b}'} \asymp Id_{\mathbb{R}^n}$ and $\langle x \rangle_{z, \mathbf{b}} \leq K \langle x \rangle_{z', \mathbf{b}'}$ follows immediately. Since $x \mapsto \|\widetilde{M}_{z, x}^{\mathbf{b}} (\widetilde{M}_{z', x}^{\mathbf{b}'})^{-1}\| = \|(d\psi_{z, z'}^{\mathbf{b}', \mathbf{b}})_{n_z^{\mathbf{b}}(x)}\|$ and $x \mapsto \|M_{z, x}^{\mathbf{b}} (M_{z', x}^{\mathbf{b}'})^{-1}\| = \|(d\psi_{z, z'}^{\mathbf{b}, \mathbf{b}'})_{n_{z'}^{\mathbf{b}'}(x)}\|$ are bounded functions, (5.3) follows. The case where (M, \exp) has a \mathcal{O}_M -bounded geometry is similar.

(ii) We have for any $f \in C^\infty(M, \mathfrak{E})$,

$$\partial_{z, \mathbf{b}}^\alpha (f) = \partial^\alpha (f \circ (n_z^{\mathbf{b}})^{-1}) \circ n_z^{\mathbf{b}} = \partial^\alpha (f \circ (n_{z'}^{\mathbf{b}'})^{-1} \circ \psi_{z', z}^{\mathbf{b}', \mathbf{b}}) \circ n_z^{\mathbf{b}}.$$

We now apply the multivariate Faa di Bruno formula obtained by G.M. Constantine and T.H. Savits in [39], that we reformulated for convenience in Theorem 5.2.11. This formula entails that for any n -multi-index $\alpha \neq 0$,

$$\partial^\alpha (f \circ (n_{z'}^{\mathbf{b}'}))^{-1} \circ \psi_{z',z}^{\mathbf{b}',\mathbf{b}} = \sum_{1 \leq |\alpha'| \leq |\alpha|} P_{\alpha,\alpha'}(\psi_{z',z}^{\mathbf{b}',\mathbf{b}}) (\partial^{\alpha'} f \circ (n_{z'}^{\mathbf{b}'}))^{-1} \circ \psi_{z',z}^{\mathbf{b}',\mathbf{b}}$$

and thus

$$\partial_{z,\mathbf{b}}^\alpha = \sum_{1 \leq |\alpha'| \leq |\alpha|} (P_{\alpha,\alpha'}(\psi_{z',z}^{\mathbf{b}',\mathbf{b}}) \circ n_z^{\mathbf{b}}) \partial_{z',\mathbf{b}'}^{\alpha'} =: \sum_{1 \leq |\alpha'| \leq |\alpha|} f_{\alpha,\alpha'} \partial_{z',\mathbf{b}'}^{\alpha'}$$

where $P_{\alpha,\alpha'}(g)$ is a linear combination of terms of the form $\prod_{j=1}^s (\partial^{l^j} g)^{k^j}$, where $1 \leq s \leq |\alpha|$ and the k^j and l^j are n -multi-indices with $|k^j| > 0$, $|l^j| > 0$, $\sum_{j=1}^s |k^j| = |\alpha'|$ and $\sum_{j=1}^s |k^j| |l^j| = |\alpha|$. In the case where (M, \exp) has a S_σ -bounded geometry, for each $s, (k^j), (l^j)$, there is $K > 0$ such that for any $x \in \mathbb{R}^n$,

$$|\prod_{j=1}^s (\partial^{l^j} \psi_{z',z}^{\mathbf{b}',\mathbf{b}})^{k^j}(x)| \leq K \langle x \rangle^{-\sigma \sum_{j=1}^s (|l^j|-1)|k^j|} = K \langle x \rangle^{-\sigma(|\alpha|-|\alpha'|)}$$

which gives the result. The case where (M, \exp) has a \mathcal{O}_M -bounded geometry is similar. \square

Theorem 5.2.11. [39] *Let $f \in C^\infty(\mathbb{R}^p, \mathfrak{E})$ and $g \in C^\infty(\mathbb{R}^n, \mathbb{R}^p)$. Then for any n -multi-index $\nu \neq 0$,*

$$\partial^\nu (f \circ g) = \sum_{1 \leq |\lambda| \leq |\nu|} (\partial^\lambda f) \circ g \sum_{s=1}^{|\nu|} \sum_{p_s(\nu,\lambda)} \nu! \prod_{j=1}^s \frac{1}{k^j! (l^j!)^{|k^j|}} (\partial^{l^j} g)^{k^j}$$

where $p_s(\nu, \lambda)$ is the set of p -multi-indices k^j and n -multi-indices l^j ($1 \leq j \leq s$) such that $0 \prec l^1 \prec \dots \prec l^s$ ($l \prec l'$ being defined as “ $|l| < |l'|$ or $|l| = |l'|$ and $l <_L l'$ ” where $<_L$ is the strict lexicographical order), $|k^j| > 0$, $\sum_{j=1}^s k^j = \lambda$ and $\sum_{j=1}^s |k^j| |l^j| = \nu$.

Note that by Lemma 5.2.10, if (M, \exp) satisfies $(S_\sigma 1)$ (resp. $(\mathcal{O}_M 1)$), then $(S_\sigma 2)$ (resp. $(\mathcal{O}_M 2)$) is equivalent to: for any $z', z'' \in M$, there exists a frame (z, \mathbf{b}) such that $\partial_{z,\mathbf{b}}^\alpha \tau_{z'}^{-1} \tau_{z''}(x) = \mathcal{O}(\langle x \rangle_{z,\mathbf{b}}^{-\sigma|\alpha|})$ (resp. $\mathcal{O}(\langle x \rangle_{z,\mathbf{b}}^{p_\alpha})$ for a $p_\alpha \geq 1$) for any n -multi-index α .

As the following proposition shows, S_σ or \mathcal{O}_M -bounded geometry properties can be transported by any diffeomorphism.

Proposition 5.2.12. *If (M, \exp_M) has a S_σ (resp. \mathcal{O}_M) bounded geometry, N a smooth manifold and $\varphi : M \rightarrow N$ is a diffeomorphism, then $(N, \exp_N := \varphi \circ \exp_M \circ d\varphi^{-1})$ has a S_σ (resp. \mathcal{O}_M) bounded geometry.*

Proof. Let us note $\psi_{z,z',N}^{\mathbf{b},\mathbf{b}'} := n_{z,N}^{\mathbf{b}} \circ (n_{z',N}^{\mathbf{b}'})^{-1}$ where $n_{z,N}^{\mathbf{b}} := L_{\mathbf{b}} \circ \exp_{N,z}^{-1}$ and $(z, \mathbf{b}), (z', \mathbf{b}')$ are two frames on N . Since $\exp_{z',N} = \varphi \circ \exp_{M,\varphi^{-1}(z')} \circ (d\varphi^{-1})_{z'}$ and $\exp_{N,z}^{-1} = (d\varphi^{-1})_z^{-1} \circ \exp_{M,\varphi^{-1}(z)}^{-1} \circ \varphi^{-1}$, we obtain $\psi_{z,z',N}^{\mathbf{b},\mathbf{b}'} = \psi_{\varphi^{-1}(z),\varphi^{-1}(z'),M}^{\mathbf{b}_z,\mathbf{b}'_z}$ where \mathbf{b}_z is the basis of $T_{\varphi^{-1}(z)}(M)$ such that $L_{\mathbf{b}_z} = L_{\mathbf{b}} \circ (d\varphi)_{\varphi^{-1}(z)}$. The result follows. \square

The following technical lemma will be used for Fourier transform and the definition of rapidly decreasing section spaces over the tangent and cotangent bundle in section 3. It will also give the behaviour of symbols under coordinate change.

Lemma 5.2.13. *Let $(z, \mathbf{b}), (z', \mathbf{b}')$ be given frames.*

(i) *We can express $\partial_{z, \mathbf{b}}^{(\alpha, \beta)}$ as an operator on $C^\infty(T^*M, \mathfrak{E})$ (resp. $C^\infty(TM, \mathfrak{E})$), where (α, β) is a $2n$ -multi-index, with the following finite sum:*

$$\partial_{z, \mathbf{b}}^{(\alpha, \beta)} = \sum_{\substack{0 \leq |\alpha', \beta'| \leq |\alpha, \beta| \\ |\beta'| \geq |\beta|}} f_{\alpha, \beta, \alpha', \beta'} \partial_{z', \mathbf{b}'}^{(\alpha', \beta')}$$

where the $f_{\alpha, \beta, \alpha', \beta'}$ are smooth functions on T^*M (resp. TM) such that

(a) *if (M, \exp) has a S_σ -bounded geometry for a given $\sigma \in [0, 1]$, there exists $C_{\alpha, \beta} > 0$ such that for any $(x, \theta) \in T^*M$ (resp. TM),*

$$|f_{\alpha, \beta, \alpha', \beta'}(x, \theta)| \leq C_{\alpha, \beta} \langle x \rangle_{z, \mathbf{b}}^{\sigma(|\alpha'| - |\alpha|)} \langle \theta \rangle_{z, \mathbf{b}, x}^{|\beta'| - |\beta|}. \quad (5.6)$$

(b) *if (M, \exp) has a \mathcal{O}_M -bounded geometry, there exist $C_{\alpha, \beta} > 0$ and $q_{\alpha, \beta} \geq 1$ such that for any $(x, \theta) \in T^*M$ (resp. TM),*

$$|f_{\alpha, \beta, \alpha', \beta'}(x, \theta)| \leq C_{\alpha, \beta} \langle x \rangle_{z, \mathbf{b}}^{q_{\alpha, \beta}} \langle \theta \rangle_{z, \mathbf{b}, x}^{|\beta'| - |\beta|}. \quad (5.7)$$

(ii) *We can express $\partial_{z, \mathbf{b}}^{(\alpha, \beta)}$ as an operator on $C^\infty(M \times M, \mathfrak{E})$, with the following finite sum:*

$$\partial_{z, \mathbf{b}}^{(\alpha, \beta)} = \sum_{\substack{0 \leq |\alpha'| \leq |\alpha| \\ 0 \leq |\beta'| \leq |\beta|}} f_{\alpha, \beta, \alpha', \beta'} \partial_{z', \mathbf{b}'}^{(\alpha', \beta')}$$

where the $f_{\alpha, \beta, \alpha', \beta'}$ are smooth functions on $M \times M$ such that

(a) *if (M, \exp) has a S_σ -bounded geometry for a given $\sigma \in [0, 1]$, there exists $C_{\alpha, \beta} > 0$ such that for any $(x, y) \in M \times M$,*

$$|f_{\alpha, \beta, \alpha', \beta'}(x, y)| \leq C_{\alpha, \beta} \langle x \rangle_{z, \mathbf{b}}^{\sigma(|\alpha'| - |\alpha|)} \langle y \rangle_{z, \mathbf{b}}^{\sigma(|\beta'| - |\beta|)}. \quad (5.8)$$

(b) *if (M, \exp) has a \mathcal{O}_M -bounded geometry, there exist $C_{\alpha, \beta} > 0$ and $q_\alpha, q_\beta \geq 1$ such that for any $(x, y) \in M \times M$,*

$$|f_{\alpha, \beta, \alpha', \beta'}(x, y)| \leq C_{\alpha, \beta} \langle x \rangle_{z, \mathbf{b}}^{q_\alpha} \langle y \rangle_{z, \mathbf{b}}^{q_\beta}. \quad (5.9)$$

Proof. (i) Suppose that (M, \exp) has a S_σ -bounded geometry. Let us note $\psi_* := n_{z', *}' \circ (n_{z, *}^{\mathbf{b}})^{-1}$ and $\psi_T := n_{z', T}' \circ (n_{z, T}^{\mathbf{b}})^{-1}$. We have $\psi_* = (\psi_{z', z}^{\mathbf{b}', \mathbf{b}} \circ \pi_1, L)$ where π_1 is the projection from \mathbb{R}^{2n} onto the first copy of \mathbb{R}^n in \mathbb{R}^{2n} and L is the smooth map from \mathbb{R}^{2n} to \mathbb{R}^n defined as $L(x, \vartheta) := {}^t(d\psi_{z', z}^{\mathbf{b}', \mathbf{b}})^{-1}(\vartheta) = {}^t(d\psi_{z, z'}^{\mathbf{b}, \mathbf{b}'})_{\psi_{z', z}^{\mathbf{b}', \mathbf{b}}(x)}(\vartheta)$. Noting $(L_i)_{1 \leq i \leq n}$ the components of L , we have $L_i(x, \vartheta) = \sum_{1 \leq p \leq n} L_{i, p}(x) \vartheta_p$, where $L_{i, p} := (\partial_i \psi_{z, z'}^{\mathbf{b}, \mathbf{b}'})_p \circ \psi_{z', z}^{\mathbf{b}', \mathbf{b}}$. As a consequence, for $1 \leq i \leq n$ and α, β , n -multi-indices such that $|(\alpha, \beta)| > 0$

$$(\partial^{(\alpha, \beta)} \psi_*)_i = \delta_{\beta, 0} (\partial^\alpha \psi_{z', z}^{\mathbf{b}', \mathbf{b}})_i \circ \pi_1, \quad (\partial^{(\alpha, \beta)} \psi_*)_{n+i} = (\partial^{(\alpha, \beta)} L)_i,$$

$$(\partial^{(\alpha, \beta)} L)_i(x, \vartheta) = \sum_{1 \leq p \leq n} (\partial^\alpha L_{i, p})(x) F_{\beta, p}(\vartheta),$$

$$\partial^\alpha L_{i, p} = \sum_{1 \leq |\alpha'| \leq |\alpha|} P_{\alpha, \alpha'}(\psi_{z', z}^{\mathbf{b}', \mathbf{b}}) ((\partial^{\alpha' + e_i} \psi_{z, z'}^{\mathbf{b}, \mathbf{b}'})_p \circ \psi_{z', z}^{\mathbf{b}', \mathbf{b}}) \quad \text{if } |\alpha| > 0,$$

where $F_{\beta,p}(\vartheta)$ is equal to ϑ_p if $\beta = 0$, to $\delta_{p,r}$ if $\beta = e_r$, and to 0 otherwise. We get from the proof of Lemma 5.2.10 that (for $1 \leq |\alpha'| \leq |\alpha|$) $P_{\alpha,\alpha'}(\psi_{z',z}^{\mathbf{b}',\mathbf{b}})(\mathbf{x}) = \mathcal{O}(\langle \mathbf{x} \rangle^{-\sigma(|\alpha|-|\alpha'|)})$. As a consequence, using (5.2), we see that $\partial^\alpha L_{i,p}(\mathbf{x}) = \mathcal{O}(\langle \mathbf{x} \rangle^{-\sigma|\alpha|})$. Thus, if $|\beta| > 1$, $\partial^{(\alpha,\beta)}\psi_* = 0$ and

$$\begin{aligned} \text{if } \beta = 0, \quad & (\partial^{(\alpha,\beta)}\psi_*)_i(\mathbf{x}, \vartheta) = \mathcal{O}(\langle \mathbf{x} \rangle^{-\sigma(|\alpha|-1)}) \quad \text{and} \quad (\partial^{(\alpha,\beta)}\psi_*)_{n+i}(\mathbf{x}, \vartheta) = \mathcal{O}(\langle \mathbf{x} \rangle^{-\sigma|\alpha|}\langle \vartheta \rangle), \\ \text{if } |\beta| = 1, \quad & (\partial^{(\alpha,\beta)}\psi_*)_i = 0 \quad \text{and} \quad (\partial^{(\alpha,\beta)}\psi_*)_{n+i}(\mathbf{x}, \vartheta) = \mathcal{O}(\langle \mathbf{x} \rangle^{-\sigma|\alpha|}). \end{aligned}$$

Similar results hold for ψ_T , the only difference is that we just have to take $\tilde{L} := (d\psi_{z',z}^{\mathbf{b}',\mathbf{b}})_\mathbf{x}(\vartheta)$ instead of L .

We have for any $f \in C^\infty(T^*M, \mathfrak{E})$,

$$\partial_{z,\mathbf{b}}^\nu(f) = \partial^\nu(f \circ (n_{z,*}^{\mathbf{b}})^{-1}) \circ n_{z,*}^{\mathbf{b}} = \partial^\nu(f \circ (n_{z',*}^{\mathbf{b}'})^{-1} \circ \psi_*) \circ n_{z,*}^{\mathbf{b}}.$$

Using again the Faa di Bruno formula in Theorem 5.2.11, we get

$$\partial_{z,\mathbf{b}}^\nu = \sum_{1 \leq |\nu'| \leq |\nu|} (P_{\nu,\nu'}(\psi_*) \circ n_{z,*}^{\mathbf{b}}) \partial_{z',\mathbf{b}'}^{\nu'} =: \sum_{1 \leq |\nu'| \leq |\nu|} f_{\nu,\nu'} \partial_{z',\mathbf{b}'}^{\nu'}$$

where $P_{\nu,\nu'}(\psi_*)$ is a linear combination of terms of the form $\prod_{j=1}^s (\partial^{l^j}\psi_*)^{k^j}$, where $1 \leq s \leq |\nu|$, the k^j and l^j are $2n$ -multi-indices with $|k^j| > 0$, $|l^j| > 0$, $\sum_{j=1}^s k^j = \nu'$ and $\sum_{j=1}^s |k^j|l^j = \nu$.

Let us note $l^j =: (l^{j,1}, l^{j,2})$, $k^j =: (k^{j,1}, k^{j,2})$ where $l^{j,1}, l^{j,2}, k^{j,1}, k^{j,2}$ are n -multi-indices. Thus,

$$(\partial^{l^j}\psi_*)^{k^j} = \prod_{i=1}^n ((\partial^{l^j}\psi_*)_i)^{k_i^{j,1}} ((\partial^{l^j}\psi_*)_{n+i})^{k_i^{j,2}}$$

and we get, for a given s , (l^j) , (k^j) such that $(\partial^{l^j}\psi_*)^{k^j} \neq 0$ for all $1 \leq j \leq s$,

$$\begin{aligned} \text{if } l^{j,2} = 0, \quad & (\partial^{l^j}\psi_*)^{k^j} = \mathcal{O}(\langle \mathbf{x} \rangle^{-\sigma(|l^j|-1)|k^j|-\sigma|k^{j,2}|}\langle \vartheta \rangle^{|k^{j,2}|}), \\ \text{if } |l^{j,2}| = 1, \quad & k^{j,1} = 0 \text{ and } (\partial^{l^j}\psi_*)^{k^j} = \mathcal{O}(\langle \mathbf{x} \rangle^{-\sigma(|l^j|-1)|k^j|}). \end{aligned}$$

Since $k^j \neq 0$ and $(\partial^{l^j}\psi_*)^{k^j} \neq 0$, $l^{j,2}$ always satisfies $|l^{j,2}| \leq 1$. By permutation on the j indices, we can suppose that for $1 \leq j \leq j_1 - 1$, we have $l^{j,2} = 0$, for $j_1 \leq j \leq s$, we have $|l^{j,2}| = 1$, where $1 \leq j_1 \leq s + 1$. Thus,

$$\prod_{j=1}^s (\partial^{l^j}\psi_*)^{k^j} = \mathcal{O}(\langle \mathbf{x} \rangle^{-\sigma(\sum_{j=1}^s (|l^j|-1)|k^j| + \sum_{j=1}^{j_1-1} |k^{j,2}|)} \langle \vartheta \rangle^{\sum_{j=1}^{j_1-1} |k^{j,2}|}).$$

Since, with $\nu = (\alpha, \beta)$, $\nu' = (\alpha', \beta')$,

$$\sum_{j=1}^{j_1-1} |k^{j,2}| = \sum_{j=1}^s |k^{j,2}| - \sum_{j=j_1}^s |k^{j,2}| = |\beta'| - \sum_{j=j_1}^s |k^j||l^{j,2}| = |\beta'| - |\beta|,$$

(5.6) follows. If we set $f_{0,0,0,0} := 1$ and $f_{\alpha,0,0,0} := 0$ if $\alpha \neq 0$, then for any $2n$ -multi-index (α, β) ,

$$\partial_{z,\mathbf{b}}^{(\alpha,\beta)} = \sum_{\substack{0 \leq |(\alpha',\beta')| \leq |(\alpha,\beta)| \\ |\beta'| \geq |\beta|}} f_{\alpha,\beta,\alpha',\beta'} \partial_{z',\mathbf{b}'}^{(\alpha',\beta')}$$

and the estimate (5.6) holds for any $f_{\alpha,\beta,\alpha',\beta'}$. In the case of \mathcal{O}_M -bounded geometry, the proof is similar, and we obtain for a $r_\nu \geq 1$, $\prod_{j=1}^s (\partial^{l^j}\psi_*)^{k^j} = \mathcal{O}(\langle \mathbf{x} \rangle^{r_\nu} \langle \vartheta \rangle^{|\beta'|-|\beta|})$, which gives the result.

(ii) Replacing ψ_* by $\psi_{z',z,M^2}^{\mathbf{b}',\mathbf{b}} := n_{z',M^2}^{\mathbf{b}'} \circ (n_{z,M^2}^{\mathbf{b}})^{-1}$ in (i), we obtain the result by similar arguments. \square

5.2.4 Basic function and distribution spaces

We suppose in this section that E is an hermitian vector bundle on the exponential manifold (M, \exp) . Recall that if $u \in C^\infty(M; E)$ (resp. $C_c^\infty(M; E)$) the Fréchet space of smooth sections (resp. the LF -space of compactly supported smooth sections) of $E \rightarrow M$, we have for any $z \in M$, $u^z := \tau_z^{-1}u \in C^\infty(M, E_z)$ (resp. $C_c^\infty(M, E_z)$). We define for any frame (z, \mathbf{b}) on M ,

$$T_{z, \mathbf{b}}(u) := u^z \circ (n_z^{\mathbf{b}})^{-1}.$$

Thus, $T_{z, \mathbf{b}}$ sends sections of $E \rightarrow M$ to functions from \mathbb{R}^n to E_z and is in fact a topological isomorphism from $C^\infty(M; E)$ (resp. $C_c^\infty(M; E)$) onto $C^\infty(\mathbb{R}^n, E_z)$ (resp. $C_c^\infty(\mathbb{R}^n, E_z)$).

In the following, a density (resp. a codensity) is a smooth section of the complex line bundle over M defined by the disjoint union over $x \in M$ of the complex lines formed by the 1-twisted forms on $T_x M$ (resp. $T_x^*(M)$). Recall that a 1-twisted form on a n -dimensional vector space V is a function on F on $\Lambda_n V \setminus \{0\}$ such that

$$F(cv) = |c|F(v) \quad \text{for all } v \in \Lambda_n V \setminus \{0\} \text{ and } c \in \mathbb{R}^*.$$

For a given frame (z, \mathbf{b}) , let us note $|dx^{z, \mathbf{b}}|$ the density associated to the volume form on M : $dx^{z, \mathbf{b}} := dx^{1, z, \mathbf{b}} \wedge \dots \wedge dx^{n, z, \mathbf{b}}$ and $|\partial_{z, \mathbf{b}}|$ the codensity defined as $|\partial_{1, z, \mathbf{b}} \wedge \dots \wedge \partial_{n, z, \mathbf{b}}|$.

Any density (resp. codensity) is of the form $c|dx^{z, \mathbf{b}}|$ (resp. $c|\partial_{z, \mathbf{b}}|$) where c is a smooth function on M , and by definition is strictly positive if $c(x) > 0$ for any $x \in M$. For a given strictly positive density $d\mu$, we also note by $d\mu$ its associated (positive, Borel–Radon, σ -finite) measure on M . This allows to define the following Banach spaces of (equivalence classes of) functions on M : $L^p(M, d\mu)$ ($1 \leq p \leq \infty$). Actually, $L^\infty(M) := L^\infty(M, d\mu)$ does not depend on the chosen $d\mu$, since the null sets for $d\mu$ are exactly the null sets for any other strictly positive density $d\mu'$ on M .

For a given $z \in M$, we denote $L^p(M, E_z, d\mu)$ ($1 \leq p < \infty$) and $L^\infty(M, E_z)$ the Bochner spaces on M with values in E_z . E_z is a hermitian complex vector space, so we can identify E_z with its antidual E'_z . There is a natural anti-isomorphism between E'_z and the dual of E_z but there is in general no canonical way to identify E_z with its dual with a *linear* isomorphism. Thus, we shall use antiduals rather than duals in the following. However, E_z is anti-isomorphic with its dual by complex conjugaison on E'_z . We shall note \bar{x} the image under this anti-isomorphism of $x \in E_z$ and \bar{E}_z the dual of E_z .

We denote $L^p(M; E, d\mu) := \{\psi \text{ section of } E \rightarrow M \text{ such that } |\psi|^p \in L^1(M, d\mu)\} / \sim_{a.e.}$ and $L^\infty(M; E) := \{\psi \text{ section of } E \rightarrow M \text{ such that } |\psi| \in L^\infty(M)\} / \sim_{a.e.}$ where $\sim_{a.e.}$ the standard “almost everywhere” equivalence relation. Since the τ_{xy} maps are isometries, for any $z \in M$, the map $\psi \rightarrow \tau_z^{-1}\psi$ defines linear isometries: $L^p(M; E, d\mu) \simeq L^p(M, E_z, d\mu)$, and $L^\infty(M; E) \simeq L^\infty(M, E_z)$. In particular, $L^p(M; E, d\mu)$ and $L^\infty(M; E)$ are Banach spaces and $L^2(M; E, d\mu)$ a Hilbert space. Moreover, we can define for any $\psi \in L^1(M; E, d\mu)$ and $z \in M$ the following Bochner integral $\int \tau_z^{-1}\psi \in E_z$. We can canonically identify $L^\infty(M; E)$ as the antidual of $L^1(M; E, d\mu)$ and $L^2(M; E, d\mu)$ as its own antidual. The (strong) antiduals of $C_c^\infty(M; E)$ and $C^\infty(M; E)$ are noted respectively $\mathcal{D}'(M; E)$ and $\mathcal{E}'(M; E)$.

We define $G_\sigma(\mathbb{R}^p, \mathfrak{E})$ (resp. $S_\sigma(\mathbb{R}^p)$), where $\sigma \in [0, 1]$, as the space of smooth functions g from \mathbb{R}^p into \mathfrak{E} (resp. \mathbb{R}) such that for any p -multi-index $\nu \neq 0$ (resp. any p -multi-index ν), there exists $C_\nu > 0$ such that $\|\partial^\nu g(\mathbf{v})\| \leq C_\nu \langle \mathbf{v} \rangle^{-\sigma(|\nu|-1)}$ (resp. $|\partial^\nu g(\mathbf{v})| \leq C_\nu \langle \mathbf{v} \rangle^{-\sigma|\nu|}$) for any $\mathbf{v} \in \mathbb{R}^p$. We note $\mathcal{O}_M(\mathbb{R}^p, \mathfrak{E})$ the space of smooth \mathfrak{E} -valued functions with polynomially bounded derivatives. We use the shortcuts $G_\sigma(\mathbb{R}^p) := G_\sigma(\mathbb{R}^p, \mathbb{R})$ and $\mathcal{O}_M(\mathbb{R}^p) := \mathcal{O}_M(\mathbb{R}^p, \mathbb{R})$.

We have the following lemma which will give an equivalent formulation of S_σ or \mathcal{O}_M -bounded geometry.

Lemma 5.2.14. (i) Let $f \in G_\sigma(\mathbb{R}^p, \mathfrak{E})$ (resp. $S_\sigma(\mathbb{R}^p)$) and $g \in G_\sigma(\mathbb{R}^n, \mathbb{R}^p)$ such that, if $\sigma > 0$, there exists $\varepsilon > 0$ such that $\langle g(v) \rangle \geq \varepsilon \langle v \rangle$ for any $v \in \mathbb{R}^n$. Then $f \circ g \in G_\sigma(\mathbb{R}^n, \mathfrak{E})$ (resp. $S_\sigma(\mathbb{R}^n)$).
(ii) The set $G_\sigma^\times(\mathbb{R}^p)$ of diffeomorphisms g on \mathbb{R}^p such that g and g^{-1} are in $G_\sigma(\mathbb{R}^p)$ is a subgroup of $\text{Diff}(\mathbb{R}^p)$ and contains $GL_p(\mathbb{R})$ as a subgroup.
(iii) We have $\mathcal{O}_M(\mathbb{R}^p, \mathfrak{E}) \circ \mathcal{O}_M(\mathbb{R}^n, \mathbb{R}^p) \subseteq \mathcal{O}_M(\mathbb{R}^n, \mathfrak{E})$. In particular, the space $\mathcal{O}_M(\mathbb{R}^p, \mathbb{R}^p)$ is a monoid under the composition of functions. The set of invertible elements of the monoid $\mathcal{O}_M(\mathbb{R}^p, \mathbb{R}^p)$, noted $\mathcal{O}_M^\times(\mathbb{R}^p, \mathbb{R}^p)$, is a subgroup of $\text{Diff}(\mathbb{R}^p)$ and contains $G_\sigma^\times(\mathbb{R}^p)$ as a subgroup.
(iv) (M, \exp) has a S_σ (resp. \mathcal{O}_M)-bounded geometry if and only if there exists a frame (z_0, \mathbf{b}_0) such that for any frame (z, \mathbf{b}) , $\psi_{z_0, z}^{\mathbf{b}_0, \mathbf{b}} \in G_\sigma^\times(\mathbb{R}^n)$ (resp. $\mathcal{O}_M^\times(\mathbb{R}^n, \mathbb{R}^n)$).
(v) The set, noted $S_\sigma^\times(\mathbb{R}^p)$ (resp. $\mathcal{O}_M^\times(\mathbb{R}^p)$), of smooth functions $f : \mathbb{R}^p \rightarrow \mathbb{R}^*$ such that f and $1/f$ are in $S_\sigma(\mathbb{R}^p)$ (resp. $\mathcal{O}_M(\mathbb{R}^p)$) is a commutative group under pointwise multiplication of functions. Moreover, $S_\sigma^\times(\mathbb{R}^p) \leq S_{\sigma'}^\times(\mathbb{R}^p) \leq \mathcal{O}_M^\times(\mathbb{R}^p)$ if $1 \geq \sigma \geq \sigma' \geq 0$.
(vi) If $g \in G_\sigma^\times(\mathbb{R}^p)$ (resp. $\mathcal{O}_M^\times(\mathbb{R}^p, \mathbb{R}^p)$) then its Jacobian determinant $J(g)$ is in $S_\sigma^\times(\mathbb{R}^p)$ (resp. $\mathcal{O}_M^\times(\mathbb{R}^p)$).

Proof. (i) The Faa di Bruno formula yields for any n -multi-index $\nu \neq 0$,

$$\partial^\nu(f \circ g) = \sum_{1 \leq |\lambda| \leq |\nu|} (\partial^\lambda f) \circ g P_{\nu, \lambda}(g) \quad (5.10)$$

where $P_{\nu, \lambda}(g)$ is a linear combination (with coefficients independent on f and g) of functions of the form $\prod_{j=1}^s (\partial^{k^j} g)^{l^j}$ where $s \in \{1, \dots, |\nu|\}$. The k^j are p -multi-indices and the l^j are n -multi-indices (for $1 \leq j \leq s$) such that $|k^j| > 0$, $|l^j| > 0$, $\sum_{j=1}^s k^j = \lambda$ and $\sum_{j=1}^s |k^j| l^j = \nu$. As a consequence, since $g \in G_\sigma(\mathbb{R}^n, \mathbb{R}^p)$, for each ν, λ with $1 \leq |\lambda| \leq |\nu|$ there exists $C_{\nu, \lambda} > 0$ such that for any $v \in \mathbb{R}^n$,

$$|P_{\nu, \lambda}(g)(v)| \leq C_{\nu, \lambda} \langle v \rangle^{-\sigma(|\nu| - |\lambda|)}. \quad (5.11)$$

Moreover, if $f \in G_\sigma(\mathbb{R}^p, \mathfrak{E})$ (resp. $S_\sigma(\mathbb{R}^p)$), there is $C'_\lambda > 0$ such that for any $v \in \mathbb{R}^n$, $\|(\partial^\lambda f) \circ g(v)\| \leq C'_\lambda \langle v \rangle^{-\sigma(|\lambda| - 1)}$ (resp. $|(\partial^\lambda f) \circ g(v)| \leq C'_\lambda \langle v \rangle^{-\sigma|\lambda|}$). The result now follows from (5.10) and (5.11).

(ii) Let f and g in $G_\sigma^\times(\mathbb{R}^p)$. We have $\partial_i g^{-1} = \mathcal{O}(1)$ for any $i \in \{1, \dots, p\}$. Taylor–Lagrange inequality of order 1 entails that $\langle g^{-1}(v) \rangle = \mathcal{O}(\langle v \rangle)$ and thus there is $\varepsilon > 0$ such that $\langle g(v) \rangle \geq \varepsilon \langle v \rangle$ for any $v \in \mathbb{R}^n$. With (i), we get $f \circ g \in G_\sigma(\mathbb{R}^p)$. The same argument shows that $g^{-1} \circ f^{-1} \in G_\sigma(\mathbb{R}^p)$.

(iii) Direct consequence of Theorem 5.2.11.

(iv) The only if part is obvious. Suppose then that for any frame (z, \mathbf{b}) , $\psi_{z_0, z}^{\mathbf{b}_0, \mathbf{b}} \in G_\sigma^\times(\mathbb{R}^n)$ (resp. $\mathcal{O}_M^\times(\mathbb{R}^n, \mathbb{R}^n)$). Let (z, \mathbf{b}) , (z', \mathbf{b}') be two frames. We have $\psi_{z, z'}^{\mathbf{b}, \mathbf{b}'} = \psi_{z, z_0}^{\mathbf{b}, \mathbf{b}_0} \circ \psi_{z_0, z'}^{\mathbf{b}_0, \mathbf{b}'}$. So, by (ii) (resp. (iii)), $\psi_{z, z'}^{\mathbf{b}, \mathbf{b}'} \in G_\sigma^\times(\mathbb{R}^n)$ (resp. $\mathcal{O}_M^\times(\mathbb{R}^n, \mathbb{R}^n)$), which yields the result.

(v) By Leibniz rule, the spaces $S_\sigma(\mathbb{R}^p)$ and $\mathcal{O}_M(\mathbb{R}^p)$ are \mathbb{R} -algebras under the pointwise product of functions. The result follows.

(vi) Consequence of (ii), (iii), $1/J(g) = J(g^{-1}) \circ g$ and the fact that $S_\sigma(\mathbb{R}^p)$ (resp. $\mathcal{O}_M(\mathbb{R}^p)$) is stable under the pointwise product of functions. \square

Remark that for any $g \in G_\sigma^\times(\mathbb{R}^p)$, we have $g \asymp \text{Id}_{\mathbb{R}^p}$. The multiplication by a function in $\mathcal{O}_M^\times(\mathbb{R}^n)$ is a topological isomorphism from the Fréchet space of rapidly decaying E_z -valued functions $\mathcal{S}(\mathbb{R}^n, E_z)$ onto itself. If we denote $J_{z,z'}^{\mathbf{b},\mathbf{b}'}$ the Jacobian of $\psi_{z,z'}^{\mathbf{b},\mathbf{b}'}$, then $1/J_{z,z'}^{\mathbf{b},\mathbf{b}'} = J_{z',z}^{\mathbf{b}',\mathbf{b}} \circ \psi_{z,z'}^{\mathbf{b},\mathbf{b}'}$ and $J_{z,z'}^{\mathbf{b},\mathbf{b}'} \circ n_{z'}^{\mathbf{b}'}(x) = dx^{z,\mathbf{b}}/dx^{z',\mathbf{b}'}(x) = \det M_{z,x}^{\mathbf{b}}(M_{z',x}^{\mathbf{b}'})^{-1} = \det(M_{z',x}^{\mathbf{b}'})^{-1} M_{z,x}^{\mathbf{b}}$. We deduce from Lemma 5.2.14 (vi) that if (M, \exp) has a S_σ (resp. \mathcal{O}_M) bounded geometry then $J_{z,z'}^{\mathbf{b},\mathbf{b}'}$ is in $S_\sigma^\times(\mathbb{R}^n)$ (resp. $\mathcal{O}_M^\times(\mathbb{R}^n)$).

Definition 5.2.15. Any smooth function f is in S_σ (resp. \mathcal{O}_M) if for any frame (z, \mathbf{b}) , $f \circ (n_z^{\mathbf{b}})^{-1} \in S_\sigma(\mathbb{R}^n)$ (resp. $\mathcal{O}_M(\mathbb{R}^n)$). Similarly, any smooth function f is in S_σ^\times (resp. \mathcal{O}_M^\times) if for any frame (z, \mathbf{b}) , $f \circ (n_z^{\mathbf{b}})^{-1} \in S_\sigma^\times(\mathbb{R}^n)$ (resp. $\mathcal{O}_M^\times(\mathbb{R}^n)$).

Lemma 5.2.16. If (M, \exp) has a S_σ -bounded geometry then a smooth function f on M is in S_σ (resp. S_σ^\times) if there exists a frame (z, \mathbf{b}) such that $f \circ (n_z^{\mathbf{b}})^{-1} \in S_\sigma(\mathbb{R}^n)$ (resp. $S_\sigma^\times(\mathbb{R}^n)$). Similarly, If (M, \exp) has a \mathcal{O}_M -bounded geometry then f is in \mathcal{O}_M (resp. \mathcal{O}_M^\times) if there exists a frame (z, \mathbf{b}) such that $f \circ (n_z^{\mathbf{b}})^{-1} \in \mathcal{O}_M(\mathbb{R}^n)$ (resp. $\mathcal{O}_M^\times(\mathbb{R}^n)$).

Proof. Let (z', \mathbf{b}') be a frame such that $f \circ (n_{z'}^{\mathbf{b}'})^{-1} \in S_\sigma(\mathbb{R}^n)$, and let (z, \mathbf{b}) be another frame. By Lemma 5.2.10 (ii), if (M, \exp) has a S_σ -bounded geometry then for any n -multi-index α ,

$$\partial^\alpha (f \circ (n_z^{\mathbf{b}})^{-1}) = \sum_{0 \leq |\alpha'| \leq |\alpha|} f_{\alpha, \alpha'} \circ (n_z^{\mathbf{b}})^{-1} (\partial^{\alpha'} f \circ (n_{z'}^{\mathbf{b}'})^{-1}) \circ \psi_{z', z}^{\mathbf{b}', \mathbf{b}}$$

where $(f_{\alpha, \alpha'} \circ (n_z^{\mathbf{b}})^{-1})(x) = \mathcal{O}(\langle x \rangle^{-\sigma(|\alpha| - |\alpha'|)})$. As a consequence $\partial^\alpha (f \circ (n_z^{\mathbf{b}})^{-1})(x) = \mathcal{O}(\langle x \rangle^{-\sigma|\alpha|})$ and the result follows. The case of \mathcal{O}_M bounded geometry is similar. \square

Definition 5.2.17. A smooth strictly positive density $d\mu$ is a S_σ^\times -density (resp. \mathcal{O}_M^\times -density) if for any frame (z, \mathbf{b}) , the unique smooth strictly positive function $f_{z, \mathbf{b}}$ such that $d\mu = f_{z, \mathbf{b}} |dx^{z, \mathbf{b}}|$ is in S_σ^\times (resp. \mathcal{O}_M^\times). In this case, we shall note $\mu_{z, \mathbf{b}}$ the smooth strictly positive function in $S_\sigma^\times(\mathbb{R}^n)$ (resp. $\mathcal{O}_M^\times(\mathbb{R}^n)$) such that $d\mu = (\mu_{z, \mathbf{b}} \circ n_z^{\mathbf{b}}) |dx^{z, \mathbf{b}}|$.

We shall say that $(M, \exp, d\mu)$ has a S_σ (resp. \mathcal{O}_M) bounded geometry if (M, \exp) has a S_σ (resp. \mathcal{O}_M) bounded geometry and $d\mu$ is a S_σ^\times (resp. \mathcal{O}_M^\times) density.

Lemma 5.2.18. If (M, \exp) has a S_σ (resp. \mathcal{O}_M) bounded geometry then any density of the form $u \circ n_{z'}^{\mathbf{b}'} |dx^{z, \mathbf{b}}|$ where u is a smooth strictly positive function in $S_\sigma^\times(\mathbb{R}^n)$ (resp. $\mathcal{O}_M^\times(\mathbb{R}^n)$) and (z, \mathbf{b}) , (z', \mathbf{b}') are frames, is a S_σ^\times -density (resp. \mathcal{O}_M^\times -density).

Proof. Let (z'', \mathbf{b}'') be an arbitrary frame. Noting $d\mu := u \circ n_{z'}^{\mathbf{b}'} |dx^{z, \mathbf{b}}|$, we get $d\mu = (u \circ n_{z'}^{\mathbf{b}'}) |J_{z, z'}^{\mathbf{b}, \mathbf{b}'}| \circ n_{z''}^{\mathbf{b}''} |dx^{z'', \mathbf{b}''}|$. We already saw that the function $J_{z, z'}^{\mathbf{b}, \mathbf{b}'}$ is in $S_\sigma^\times(\mathbb{R}^n)$ (resp. $\mathcal{O}_M^\times(\mathbb{R}^n)$). By Lemma 5.2.16, $(u \circ n_{z'}^{\mathbf{b}'}) (|J_{z, z'}^{\mathbf{b}, \mathbf{b}'}| \circ n_{z''}^{\mathbf{b}''})$ is in S_σ^\times (resp. \mathcal{O}_M^\times). \square

Remark 5.2.19. By taking $u := x \mapsto 1$ in the previous lemma, we see that for any exponential manifold (M, \exp) with S_σ (resp. \mathcal{O}_M) bounded geometry, we can define a canonical family of S_σ^\times -densities (resp. \mathcal{O}_M^\times -densities) on M : $\mathcal{D} := (|dx^{z, \mathbf{b}}|)_{(z, \mathbf{b}) \in I}$ where I is the set of frames on M . If the map \exp is the exponential map associated to a pseudo-Riemannian metric g on M , we can also define a canonical subfamily of \mathcal{D} by $\mathcal{D}_g := (|dx^z|)_{z \in M}$ where $|dx^z| := |dx^{z, \mathbf{b}}|$ with \mathbf{b} any orthonormal basis (in the sense $g_z(\mathbf{b}_i, \mathbf{b}_j) = \eta_i \delta_{i,j}$ where $\eta_i = 1$ for $1 \leq i \leq m$ and $\eta_i = -1$ for $i > m$, where g has signature $(m, n - m)$) of $T_z(M)$ ($|dx^z|$ is then independent of \mathbf{b}). A priori, the Riemannian density does not belong to the canonical M -indexed family \mathcal{D}_g .

We shall need integrations over tangent and cotangent fibers and manifolds. We thus define $d\mu^* := (\mu_{z,\mathbf{b}}^{-1} \circ n_z^{\mathbf{b}}) |\partial_{z,\mathbf{b}}|$ the codensity associated to $d\mu$, where $\mu_{z,\mathbf{b}}^{-1} := \frac{1}{\mu_{z,\mathbf{b}}}$ and (z, \mathbf{b}) is a frame. Note that since $|\partial_{z,\mathbf{b}}|/|\partial_{z',\mathbf{b}'}| = |dx^{z',\mathbf{b}'}|/|dx^{z,\mathbf{b}}| = (\mu_{z,\mathbf{b}} \circ n_z^{\mathbf{b}})/(\mu_{z',\mathbf{b}'} \circ n_{z'}^{\mathbf{b}'})$, $d\mu^*$ is independent of (z, \mathbf{b}) . For a given $x \in M$, the density on $T_x(M)$ associated to $d\mu$ is $d\mu_x := (\mu_{z,\mathbf{b}} \circ n_z^{\mathbf{b}}(x)) |dx_x^{z,\mathbf{b}}|$ and the associated density on $T_x^*(M)$ is $d\mu_x^* := (\mu_{z,\mathbf{b}}^{-1} \circ n_z^{\mathbf{b}}(x)) |\partial_{z,\mathbf{b}}|$. For a function f defined on $T_x(M)$ or $T_x^*(M)$, we have formally:

$$\begin{aligned} \int_{T_x(M)} f(\xi) d\mu_x(\xi) &= \mu_{z,\mathbf{b}} \circ n_z^{\mathbf{b}}(x) \int_{\mathbb{R}^n} f \circ (M_{z,x}^{\mathbf{b}})^{-1}(\zeta) d\zeta, \\ \int_{T_x^*(M)} f(\theta) d\mu_x^*(\theta) &= \mu_{z,\mathbf{b}}^{-1} \circ n_z^{\mathbf{b}}(x) \int_{\mathbb{R}^n} f \circ (\widetilde{M}_{z,x}^{\mathbf{b}})^{-1}(\vartheta) d\vartheta, \end{aligned}$$

and it is straightforward to check that these integrals are independent of the chosen frame (z, \mathbf{b}) .

5.2.5 Schwartz spaces and operators

Assumption 5.2.20. *We suppose in this section and in section 5.2.6 that $(M, \exp, E, d\mu)$, where E is an hermitian vector bundle on M , has a \mathcal{O}_M -bounded geometry.*

The main consequence of the exponential structure is the possibility to define Schwartz functions on M .

Definition 5.2.21. A section $u \in C^\infty(M, E)$ is rapidly decaying at infinity if for any (z, \mathbf{b}) , any n -multi-index α and $p \in \mathbb{N}$, there exists $K_{\alpha,p} > 0$ such that the following estimate

$$\|\partial_{z,\mathbf{b}}^\alpha u^z(x)\|_{E_z} < K_{\alpha,p} \langle x \rangle_{z,\mathbf{b}}^{-p} \quad (5.12)$$

holds uniformly in $x \in M$. We note $\mathcal{S}(M, E)$ the space of such sections.

With the hypothesis of \mathcal{O}_M -bounded geometry, we see that the requirement “any (z, \mathbf{b}) ” can be reduced to a simple existence:

Lemma 5.2.22. *A section $u \in C^\infty(M, E)$ is in $\mathcal{S}(M, E)$ if and only if there exists a frame (z, \mathbf{b}) such that (5.12) is valid.*

Proof. Suppose that (5.12) is valid for (z', \mathbf{b}') and let (z, \mathbf{b}) be another frame. Thus, with Lemma 5.2.10 (ii) and Leibniz rule,

$$\partial_{z,\mathbf{b}}^\alpha u^z(x) = \sum_{0 \leq |\alpha'| \leq |\alpha|} \sum_{\beta \leq \alpha'} f_{\alpha,\alpha'} \binom{\alpha'}{\beta} \partial_{z',\mathbf{b}'}^{\alpha'-\beta} (\tau_z^{-1} \tau_{z'}) \partial_{z',\mathbf{b}'}^\beta u^{z'}(x). \quad (5.13)$$

Moreover $|f_{\alpha,\alpha'} \binom{\alpha'}{\beta} \partial_{z',\mathbf{b}'}^{\alpha'-\beta} (\tau_z^{-1} \tau_{z'})| \leq C_\alpha \langle x \rangle_{z,\mathbf{b}}^{q_\alpha}$ for a $C_\alpha > 0$ and a $q_\alpha \geq 1$. Now (5.12) and (5.4) entail that for any $p \in \mathbb{N}$, there is $K > 0$ such that $\|\partial_{z,\mathbf{b}}^\alpha u^z(x)\|_{E_z} \leq K \langle x \rangle_{z,\mathbf{b}}^{-p}$. The result follows. \square

Remark 5.2.23. *Let $u \in C^\infty(M, E)$ and (z, \mathbf{b}) a frame. Then $u \in \mathcal{S}(M, E)$ if and only if $(\tau_z^{-1} u) \circ (n_z^{\mathbf{b}})^{-1} \in \mathcal{S}(\mathbb{R}^n, E_z)$. In other words, if $v \in \mathcal{S}(\mathbb{R}^n, E_z)$ then $\tau_z(v \circ n_z^{\mathbf{b}}) \in \mathcal{S}(M, E)$.*

The following lemma shows that we can define canonical Fréchet topologies on $\mathcal{S}(M, E)$.

Lemma 5.2.24. *Let (z, \mathbf{b}) a frame. Then*

(i) *The following set of semi-norms:*

$$q_{\alpha,p}(u) := \sup_{x \in M} \langle x \rangle_{z,\mathbf{b}}^p \left\| \partial_{z,\mathbf{b}}^\alpha u^z(x) \right\|_{E_z} .$$

defines a locally convex metrizable topology \mathcal{T} on $\mathcal{S}(M, E)$.

(ii) *The application $T_{z,\mathbf{b}}$ is a topological isomorphism from the space $\mathcal{S}(M, E)$ onto $\mathcal{S}(\mathbb{R}^n, E_z)$.*

(iii) *The topology \mathcal{T} is Fréchet and independent of the chosen frame (z, \mathbf{b}) .*

Proof. (i) and (ii) are obvious.

(iii) Since $T_{z,\mathbf{b}}$ is a topological isomorphism, \mathcal{T} is complete. Following the arguments of the proof of Lemma 5.2.22, we see that there is $r \in \mathbb{N}^*$ such that for any n -multi-index α and $p \in \mathbb{N}$, there exist $C_{\alpha,p} > 0$, $r_{\alpha,p} \in \mathbb{N}^*$, such for any $u \in \mathcal{S}(M, E)$,

$$q_{\alpha,p}^{(z,\mathbf{b})}(u) \leq C_{\alpha,p} \sum_{|\beta| \leq |\alpha|} q_{\beta,r_{\alpha,p}}^{(z',\mathbf{b}')} (u) .$$

The independence on (z, \mathbf{b}) follows. \square

Remark 5.2.25. *If $(M, \exp, E, d\mu)$ has a S_0 -bounded geometry, then it is possible to define the Fréchet space of smooth sections with bounded derivatives $\mathcal{B}(M, E)$ by following the same procedure of $\mathcal{S}(M, E)$, with Lemma 5.2.10.*

Classical results of distribution theory [139] and the previous topological isomorphisms $T_{z,\mathbf{b}}$ entail the following diagrams of continuous linear injections ($(M; E)$ omitted and $1 \leq p < \infty$):

$$\begin{array}{ccccc} C_c^\infty & \longrightarrow & \mathcal{S} & \longrightarrow & C^\infty \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{E}' & \longrightarrow & \mathcal{S}' & \longrightarrow & \mathbb{D}' \end{array} \quad \begin{array}{ccccc} & & \mathcal{B} & \longrightarrow & L^\infty \\ & \nearrow & & & \searrow \\ \mathcal{S} & \longrightarrow & L^p(d\mu) & \longrightarrow & \mathcal{S}' . \end{array}$$

The injections $\mathcal{S} \rightarrow \mathcal{B} \rightarrow L^\infty$ are valid in the case where M has a S_0 -bounded geometry. In the case of a general \mathcal{O}_M -bounded geometry, only the injection $\mathcal{S} \rightarrow L^\infty$ holds a priori. The injection from functions into distribution spaces is given here by $u \mapsto \langle u, \cdot \rangle$ where $\langle u, v \rangle := \int (u|v) d\mu$. Note that the following continuous injections $\mathcal{S} \rightarrow \mathcal{S}'$ and $\mathcal{S} \rightarrow L^p(d\mu) \rightarrow \mathcal{S}'$, ($1 \leq p < \infty$) have a dense image.

Using the same principles of the definition of \mathcal{S} together with the \mathcal{O}_M -bounded geometry hypothesis and Lemma 5.2.13 (ii), we define the Fréchet space $\mathcal{S}(M \times M, L(E))$ such that for any (z, \mathbf{b}) the applications $T_{z,\mathbf{b},M^2} := K \mapsto K^z \circ (n_{z,M^2}^\mathbf{b})^{-1}$ are topological isomorphisms from $\mathcal{S}(M \times M, L(E))$ onto $\mathcal{S}(\mathbb{R}^{2n}, L(E_z))$. Noting j_{M^2} the continuous dense injection from $\mathcal{S}(M \times M, L(E))$ into its antidual $\mathcal{S}'(M \times M, L(E))$ defined as $\langle j_{M^2}(K), K' \rangle = \int_{M \times M} \text{Tr}(K(x, y)(K'(x, y))^*) d\mu \otimes d\mu(x, y)$, we have the following commutative diagram, where j is the classical continuous dense inclusion from $\mathcal{S}(\mathbb{R}^{2n}, L(E_z))$ into its antidual, and $M_{\mu \otimes \mu}$ is the multiplication operator from $\mathcal{S}(\mathbb{R}^{2n}, L(E_z))$ onto itself by the $\mathcal{O}_M^\times(\mathbb{R}^{2n})$ function $\mu_{z,\mathbf{b}} \otimes \mu_{z,\mathbf{b}}$:

$$\begin{array}{ccccc} \mathcal{S}(M \times M, L(E)) & \xrightarrow{j_{M^2}} & \mathcal{S}'(M \times M, L(E)) \\ T_{z,\mathbf{b},M^2} \downarrow & & \uparrow T_{z,\mathbf{b},M^2}^* \\ \mathcal{S}(\mathbb{R}^{2n}, L(E_z)) & \xrightarrow{M_{\mu \otimes \mu}} \mathcal{S}(\mathbb{R}^{2n}, L(E_z)) \xrightarrow{j} & \mathcal{S}'(\mathbb{R}^{2n}, L(E_z)) . \end{array}$$

Since \mathcal{S} is nuclear, $L(\mathcal{S}, \mathcal{S}') \simeq \mathcal{S}'(M \times M, L(E))$ and $\mathcal{S}(M \times M, L(E)) \simeq \mathcal{S} \hat{\otimes} \overline{\mathcal{S}}$ where $\overline{\mathcal{S}} := \mathcal{S}(M, \overline{E})$. Thus, $\mathcal{S}'(M \times M, L(E)) \simeq \mathcal{S}' \hat{\otimes} \overline{\mathcal{S}'}$, where $\overline{\mathcal{S}'}$ is the dual of \mathcal{S} which is also the antidual of $\overline{\mathcal{S}}$. Note that the isomorphism $L(\mathcal{S}, \mathcal{S}') \simeq \mathcal{S}'(M \times M, L(E))$ is given by

$$\langle A_K(v), u \rangle = K(u \otimes \bar{v})$$

where A_K is operator associated to the kernel K , $u, v \in \mathcal{S}$, and $\bar{v}(y) := \overline{v(y)}$. Formally,

$$\langle A_K(v), u \rangle = \int_{M \times M} (K(x, y)v(y)|u(x))d\mu \otimes d\mu(x, y), \quad (A_K v)(x) = \int_M K(x, y)v(y)d\mu(y).$$

Thus any continuous linear operator $A : \mathcal{S} \rightarrow \mathcal{S}'$ is uniquely determined by its kernel $K_A \in \mathcal{S}'(M \times M, L(E))$. The transfert of A into the frame (z, \mathbf{b}) is the operator $A_{z, \mathbf{b}}$ from $\mathcal{S}(\mathbb{R}^n, E_z)$ into $\mathcal{S}'(\mathbb{R}^n, E_z)$ such that

$$\langle A_{z, \mathbf{b}}(v), u \rangle := \langle A(T_{z, \mathbf{b}}^{-1}(v)), T_{z, \mathbf{b}}^{-1}(u) \rangle.$$

Thus, if K_A is the kernel of A , we have $K_{A_{z, \mathbf{b}}} := \tilde{T}_{z, \mathbf{b}, M^2}(K_A)$ as the kernel of $A_{z, \mathbf{b}}$, where $\tilde{T}_{z, \mathbf{b}, M^2}$ here is the inverse of the adjoint of T_{z, \mathbf{b}, M^2} . $\tilde{T}_{z, \mathbf{b}, M^2}$ is a topological isomorphism from $\mathcal{S}'(M \times M, L(E))$ onto $\mathcal{S}'(\mathbb{R}^{2n}, L(E_z))$.

Definition 5.2.26. An operator $A \in L(\mathcal{S}, \mathcal{S}')$ is regular if A and its adjoint A^\dagger send continously \mathcal{S} into itself. An isotropic smoothing operator is an operator with kernel in $\mathcal{S}(M \times M, L(E))$. The space regular operators and the space of isotropic smoothing operators are respectively noted $\mathfrak{R}(\mathcal{S})$ and $\Psi^{-\infty}$.

Note that this definition of isotropic smoothing operators differs from the standard smoothing operators one where only local effects are taken into account, since in this case, a smoothing operator is just an operator with smooth kernel. Clearly, A is regular if and only if for any frame (z, \mathbf{b}) , $A_{z, \mathbf{b}}$ is regular as an operator from $\mathcal{S}(\mathbb{R}^n, E_z)$ into $\mathcal{S}'(\mathbb{R}^n, E_z)$. Remark that the space of regular operators forms a $*$ -algebra under composition and the space of isotropic smoothing operators $\Psi^{-\infty}$ is a $*$ -ideal of this algebra.

Let us record the following important fact:

Proposition 5.2.27. *Any isotropic smoothing operator extends (uniquely) as a Hilbert–Schmidt operator on $L^2(d\mu)$.*

Proof. An isotropic smoothing operator A (with kernel K) extends as a continous linear operator from \mathcal{S}' to \mathcal{S} , and thus it also extends as a bounded operator on $L^2(d\mu)$. Let (z, \mathbf{b}) be a frame. If U is the unitary associated to the isomorphism $L^2(d\mu)$ onto $\mathcal{H}_{z, \mathbf{b}} := L^2(\mathbb{R}^n, E_z, \mu_{z, \mathbf{b}} dx)$ we have $A = U^* A_{z, \mathbf{b}} U$ where $A_{z, \mathbf{b}}$ is a bounded operator on $\mathcal{H}_{z, \mathbf{b}}$ given by the kernel $K^z \circ (n_z^{\mathbf{b}}, n_z^{\mathbf{b}})^{-1}$. Since this kernel is in $\mathcal{H}_{z, \mathbf{b}} \otimes \overline{\mathcal{H}}_{z, \mathbf{b}} = L^2(\mathbb{R}^{2n}, E_z \otimes \overline{E}_z, (\mu_{z, \mathbf{b}} dx)^{\otimes 2})$, it follows that $A_{z, \mathbf{b}}$ is Hilbert–Schmidt on $\mathcal{H}_{z, \mathbf{b}}$, which gives the result. \square

5.2.6 Fourier transform

Fourier transform is the fundamental element that will allow the passage from operators to their symbols. In our setting, it is natural to extend the classical Fourier transform on \mathbb{R}^n to Schwartz spaces of rapidly decreasing sections on the tangent and cotangent bundles of M , and use the fibers $T_x(M)$, $T_x^*(M)$ as support of integration.

Definition 5.2.28. A smooth section $a \in C^\infty(T^*M, L(E))$ is in $\mathcal{S}(T^*M, L(E))$ if for any (z, \mathbf{b}) , any $2n$ -multi-index ν and any $p \in \mathbb{N}$, there exists $K_{p,\nu} > 0$ such that

$$\|\partial_{z,\mathbf{b}}^\nu a^z(x, \theta)\|_{L(E_z)} \leq K_{p,\nu} \langle x, \theta \rangle_{z,\mathbf{b}}^{-p} \quad (5.14)$$

uniformly in $(x, \theta) \in T^*M$. A similar definition is set for $\mathcal{S}(TM, L(E))$.

Following the same technique as for the space $\mathcal{S}(M, E)$, using the coordinate invariance given by Lemma 5.2.13 we obtain the

Proposition 5.2.29. (i) A section $u \in C^\infty(T^*M, L(E))$ is in $\mathcal{S}(T^*M, L(E))$ if and only if there exists a frame (z, \mathbf{b}) such that (5.14) is valid. A similar property holds for $\mathcal{S}(TM, L(E))$.
(ii) There is a Fréchet topology on $\mathcal{S}(T^*M, L(E))$ such that each

$$T_{z,\mathbf{b},*} : a \mapsto a^z \circ (n_{z,*}^{\mathbf{b}})^{-1}$$

is a topological isomorphism from $\mathcal{S}(T^*M, L(E))$ onto $\mathcal{S}(\mathbb{R}^{2n}, L(E_z))$. A similar property holds for $\mathcal{S}(TM, L(E))$ and the applications $T_{z,\mathbf{b},T} := a \mapsto a^z \circ (n_{z,T}^{\mathbf{b}})^{-1}$.

Proof. (i, ii) Suppose that (5.14) is valid for (z', \mathbf{b}') and $a \in C^\infty(T^*M, L(E))$ and let (z, \mathbf{b}) another frame. With Lemma 5.2.13 and Leibniz rule, noting $\nu = (\alpha, \beta)$, $\nu' = (\alpha', \beta')$, $\lambda = (\lambda^1, \lambda^2)$ and $\rho = (\rho^1, \rho^2)$, we get

$$\partial_{z,\mathbf{b}}^\nu a^z = \sum_{\substack{0 \leq |\nu'| \leq |\nu| \\ |\beta'| \geq |\beta|}} \sum_{\rho \leq \lambda \leq \nu'} f_{\nu,\nu'} C_{\nu',\lambda,\rho} \partial_{z',\mathbf{b}'}^{\alpha' - \lambda^1} (\tau_z^{-1} \tau_{z'}) \partial_{z',\mathbf{b}'}^{(\rho^1, \beta')} (a^{z'}) \partial_{z',\mathbf{b}'}^{\lambda^1 - \rho^1} (\tau_{z'}^{-1} \tau_z) \quad (5.15)$$

where $C_{\nu',\lambda,\rho} = \delta_{\beta',\lambda^2} \delta_{\beta',\rho^2} \binom{\nu'}{\lambda} \binom{\lambda}{\rho}$. Using now the fact that for any $x, \vartheta \in \mathbb{R}^n$, $\langle x \rangle^{1/2} \langle \vartheta \rangle^{1/2} \leq \langle (x, \vartheta) \rangle \leq \langle x \rangle \langle \vartheta \rangle$, and (5.4), (5.5), we see that for any $2n$ -multi-index ν , and $p \in \mathbb{N}$, there is $r_{\nu,p} \in \mathbb{N}^*$ and $C_{\nu,p} > 0$ such that $q_{\nu,p}^{(z,\mathbf{b})}(a) \leq C_{\nu,p} \sum_{|\rho| \leq |\nu|} q_{\rho,r_{\nu,p}}^{(z',\mathbf{b}')} (a)$, where

$$q_{\nu,p}^{(z,\mathbf{b})}(a) := \sup_{(x,\theta) \in T^*M} \langle x, \theta \rangle_{z,\mathbf{b}}^p \|\partial_{z,\mathbf{b}}^\nu a^z(x, \theta)\|_{L(E_z)}.$$

The results follow, as in the case of $\mathcal{S}(M, E)$, by taking the topology given by the seminorms $q_{\nu,p}^{z,\mathbf{b}}$ for an arbitrary frame (z, \mathbf{b}) . \square

Remark 5.2.30. If (M, \exp, E) has a S_0 -bounded geometry, we saw in Remark 5.2.25 that a coordinate free (independent of the frame (z, \mathbf{b})) definition of a space of smooth E -sections on M with bounded derivatives is possible. However, a similar definition cannot be given in the same manner for $L(E)$ -sections on TM or T^*M with bounded derivatives, due to the fact that the change of coordinates of Lemma 5.2.13 impose an increasing power of $\langle \theta \rangle$ (when $|\beta'| > |\beta|$). However, the independence over (z, \mathbf{b}) would still hold for the space of smooth sections of $L(E) \rightarrow T^*M$ (resp. TM) with polynomially bounded derivatives.

We note $\mathcal{S}'(T^*M, L(E))$ and $\mathcal{S}'(TM, L(E))$ the strong antiduals of $\mathcal{S}(T^*M, L(E))$ and $\mathcal{S}(TM, L(E))$, respectively. We have the following continuous inclusion with dense image

$$j_{T^*M} : \mathcal{S}(T^*M, L(E)) \rightarrow \mathcal{S}'(T^*M, L(E)) \quad (\text{resp. } j_{TM} : \mathcal{S}(TM, L(E)) \rightarrow \mathcal{S}'(TM, L(E)))$$

defined by

$$\langle j_{T^*M}(a), b \rangle := \int_{TM^*} \text{Tr}(ab^*) d\mu^* \quad (\text{resp. } \langle j_{TM}(a), b \rangle := \int_{TM} \text{Tr}(ab^*) d\mu^T)$$

where $d\mu^*$ is the measure on T^*M given by $d\mu^*(x, \theta) := d\mu_x^*(\theta) d\mu(x)$ and $d\mu^T$ is the measure on TM given by $d\mu^T(x, \xi) := d\mu_x(\xi) d\mu(x)$. Note that for any (z, \mathbf{b}) , $d\mu^*(x, \theta) = |\partial_{z, \mathbf{b}_x}|(\theta) |dx^{z, \mathbf{b}}|(x)$ (this is the Liouville measure on T^*M) and $d\mu^T(x, \theta) = \mu_{z, \mathbf{b}}^2 \circ n_z^{\mathbf{b}}(x) |dx_x^{z, \mathbf{b}}|(\xi) |dx^{z, \mathbf{b}}|(x)$. We have the following commutative diagram, where M_{μ^2} is the multiplication operator by the $\mathcal{O}_M^\times(\mathbb{R}^{2n})$ function $(x, \zeta) \mapsto \mu_{z, \mathbf{b}}^2(x)$,

$$\begin{array}{ccc} \mathcal{S}(TM, L(E)) & \xrightarrow{j_{TM}} & \mathcal{S}'(TM, L(E)) \\ T_{z, \mathbf{b}, T} \downarrow & & \uparrow T_{z, \mathbf{b}, T}^* \\ \mathcal{S}(\mathbb{R}^{2n}, L(E_z)) & \xrightarrow{M_{\mu^2}} \mathcal{S}(\mathbb{R}^{2n}, L(E_z)) \xrightarrow{j} & \mathcal{S}'(\mathbb{R}^{2n}, L(E_z)) \end{array}$$

and, in the case of $\mathcal{S}(T^*M, L(E))$ a similar diagram is valid if M_{μ^2} is replaced by the identity.

Definition 5.2.31. The Fourier transform of $a \in \mathcal{S}(TM, L(E))$ is

$$\mathcal{F}(a) : (x, \theta) \mapsto \int_{T_x(M)} e^{-2\pi i \langle \theta, \xi \rangle} a(x, \xi) d\mu_x(\xi).$$

Proposition 5.2.32. \mathcal{F} is a topological isomorphism from $\mathcal{S}(TM, L(E))$ onto $\mathcal{S}(T^*M, L(E))$ with inverse

$$\overline{\mathcal{F}}(a) := (x, \xi) \mapsto \int_{T_x^*(M)} e^{2\pi i \langle \theta, \xi \rangle} a(x, \theta) d\mu_x^*(\theta).$$

The adjoint $\overline{\mathcal{F}}^*$ of $\overline{\mathcal{F}}$ coincides with \mathcal{F} on $\mathcal{S}(TM, L(E))$, so we still note $\overline{\mathcal{F}}^*$ by \mathcal{F} and \mathcal{F}^* by $\overline{\mathcal{F}}$.

Proof. Let (z, \mathbf{b}) be a frame. It is straightforward to check that the following diagram commutes

$$\begin{array}{ccc} \mathcal{S}(TM, L(E)) & \xrightarrow{\mathcal{F}} & \mathcal{S}(T^*M, L(E)) \\ T_{z, \mathbf{b}, T} \downarrow & & \uparrow T_{z, \mathbf{b}, *}^{-1} \\ \mathcal{S}(\mathbb{R}^{2n}, L(E_z)) & \xrightarrow{\mathcal{F}_{z, \mathbf{b}}} & \mathcal{S}(\mathbb{R}^{2n}, L(E_z)) \end{array}$$

where $\mathcal{F}_{z, \mathbf{b}} = \mathcal{F}_P \circ M_\mu = M_\mu \circ \mathcal{F}_P$, with M_μ the multiplication operator on $\mathcal{S}(\mathbb{R}^{2n}, L(E_z))$ defined by $M_\mu(a) := (x, \zeta) \mapsto \mu_{z, \mathbf{b}}(x) a(x, \zeta)$ and \mathcal{F}_P the partial Fourier transform on the space $\mathcal{S}(\mathbb{R}^{2n}, L(E_z))$ (only the variables in the second copy of \mathbb{R}^n in \mathbb{R}^{2n} being Fourier transformed). It is clear that $\mathcal{F}_{z, \mathbf{b}}$ is a topological isomorphism from $\mathcal{S}(\mathbb{R}^{2n}, L(E_z))$ onto itself with inverse $\mathcal{F}_{z, \mathbf{b}}^{-1} = M_{1/\mu} \circ \overline{\mathcal{F}}_P$. The fact that $\overline{\mathcal{F}}^*$ coincides with \mathcal{F} on $\mathcal{S}(TM, L(E))$ is a consequence of the following equality

$$\int_{TM} \text{Tr}(a(\overline{\mathcal{F}}(b))^*) d\mu^T = \int_{T^*M} \text{Tr}(\mathcal{F}(a)b^*) d\mu^*$$

for any $a \in \mathcal{S}(TM, L(E))$ and $b \in \mathcal{S}(T^*M, L(E))$, that is a direct consequence of the Parseval formula for \mathcal{F}_P . \square

5.3 Linearization and symbol maps

5.3.1 Linearization and the Φ_λ, Υ_t diffeomorphisms

Recall that a linearization (Bokobza-Haggiag [9]) on a smooth manifold M is defined as a smooth map ν from $M \times M$ into TM such that $\pi \circ \nu = \pi_1$, $\nu(x, x) = 0$ for any $x \in M$ and $(d_y \nu)_{y=x} = \text{Id}_{T_x M}$. Using this map, it is then possible by restricting ν on a small neighborhood of the diagonal of $M \times M$, to obtain a diffeomorphism onto a neighborhood of the zero section of TM and obtain an isomorphism between symbols (with a local control of the x variables on compact) and pseudodifferential operators modulo smoothing ideals. These isomorphisms depend on the linearization, as shown in [9, Proposition V.3]. We follow here the same idea, with a global point of view, since we are interested in the behavior at infinity. We thus consider, on the exponential manifold $(M, \exp, E, d\mu)$ a fixed linearization $\bar{\psi}$ that comes from an (abstract) exponential map ψ on M (also called linearization map in the following), so that $\bar{\psi}(x, y) = \psi_x^{-1}y$, and ψ_x is a diffeomorphism from $T_x M$ onto M , with $\psi_x(0) = x$, $(d\psi_x)_0 = \text{Id}_{T_x M}$. For example, ψ may be the exponential map \exp .

Let $\lambda \in [0, 1]$ and Φ_λ be the smooth map from TM onto $M \times M$ defined by

$$\Phi_\lambda : (x, \xi) \mapsto (\psi_x(\lambda\xi), \psi_x(-(1-\lambda)\xi)).$$

Assumption 5.3.1. *We suppose from now on that whenever the parameters λ, λ' , are in $]0, 1[$, it is implied that the linearization map ψ satisfies for any $x, y \in M$ and $t \in \mathbb{R}$, $\psi_x(t\psi_x^{-1}(y)) = \psi_y((1-t)\psi_y^{-1}(x))$. This hypothesis, called (H_ψ) in the following, is automatically satisfied if the linearization is derived from a exponential map of a connection on the manifold.*

A computation shows that Φ_λ is a diffeomorphism with the following inverse $\Phi_\lambda^{-1} : (x, y) \mapsto \alpha'_{yx}(1-\lambda)$ for $\lambda \neq 0$ and $\Phi_0^{-1}(x, y) := -\alpha'_{xy}(0)$, where $\alpha_{xy}(t) := \psi_x(t\psi_x^{-1}(y))$. Noting $\Phi_\lambda^{-1}(x, y) = (m_\lambda(x, y), \xi_\lambda(x, y))$, we see that $m_\lambda(x, y) = \alpha_{xy}(\lambda)$ and, if $\lambda \neq 0$, $\xi_\lambda(x, y) = \frac{1}{\lambda}\psi_{m_\lambda(x, y)}^{-1}(x)$, while $\xi_0(x, y) = -\psi_x^{-1}(y)$. In all the following, we shall use the symbol W (for Weyl) for the value $\lambda = \frac{1}{2}$, so that $m_W := m_{\frac{1}{2}}$, $\Phi_W := \Phi_{\frac{1}{2}}$, and similar conventions for the other mathematical symbols containing λ . Note that m_λ is a smooth function from $M \times M$ onto M , with $m_\lambda(x, x) = x$ for any $x \in M$. Moreover, for any $x, y \in M$, $m_\lambda(x, y) = m_{1-\lambda}(y, x)$, $m_W(x, y) = m_W(y, x)$ (the “middle point” of x and y), $\xi_\lambda(x, y) = -\xi_{1-\lambda}(y, x)$, $\xi_W(x, y) = -\xi_W(y, x)$ and $x \mapsto \Phi_\lambda^{-1}(x, x)$ is the zero section of $TM \rightarrow M$. Noting j the involution on $M \times M : (x, y) \mapsto (y, x)$, we have $\Phi_\lambda = j \circ \Phi_{1-\lambda} \circ -\text{Id}_{TM}$.

For any $t \in [-1, 1]$ (with the convention that if (H_ψ) is not satisfied, we are restricted to $t \in \{-1, 0, 1\}$), we define,

$$\Upsilon_t : (x, \xi) \mapsto (\psi_x(t\xi), \frac{-1}{t}\psi_{\psi_x(t\xi)}^{-1}(x))$$

with the convention $\frac{-1}{t}\psi_{\psi_x(t\xi)}^{-1}(x) := \xi$ if $t = 0$, so that $\Upsilon_0 = \text{Id}_{TM}$. A computation shows that $\Upsilon_t^{-1} = \Upsilon_{-t}$. The Φ_λ and Υ_t diffeomorphisms are related by the following property: for any $\lambda, \lambda' \in [0, 1]$, $\Phi_\lambda^{-1} \circ \Phi_{\lambda'} = \Upsilon_{\lambda'-\lambda}$. We will use the shorthand $\Upsilon_{t,T}(x, \xi) := \frac{-1}{t}\psi_{\psi_x(t\xi)}^{-1}(x)$, so that $\Upsilon_t = (\psi \circ t \text{Id}_{TM}, \Upsilon_{t,T})$.

Remark 5.3.2. *Note that (H_ψ) entails that $(\Upsilon_t)_{t \in \mathbb{R}}$ is a one parameter subgroup of $\text{Diff}(TM)$.*

Remark 5.3.3. Suppose that ψ is the exponential map associated to a connection ∇ on TM , and $\alpha_{x,\xi}$ the unique maximal geodesic such that $\alpha'_{x,\xi}(0) = (x, \xi)$. It is a standard result of differential geometry (see for instance [97, Theorem 3.3, p.206]) that for any $v := (x, \eta) \in TM$, and $\xi \in T_x(M)$, there exists an unique curve $\beta_v^\xi : \mathbb{R} \rightarrow TM$ such that $\nabla_{\alpha'_v} \beta_v^\xi = 0$, $\pi \circ \beta_v^\xi = \alpha_v$ (in other words, β_v^ξ is α_v -parallel lift of α_v) and $\beta_v^\xi(0) = (x, \xi)$. By definition of geodesics, $\beta_{x,\eta}^\eta = \alpha'_{x,\eta}$. Moreover, $\beta_{x,\eta}^\xi(1) \in T_{\psi_x^\eta}(M)$, so we can define the following linear isomorphism of tangent fibers: $P_{x,\eta} : T_x(M) \rightarrow T_{\psi_x^\eta}(M)$, $\xi \mapsto \beta_{x,\eta}^\xi(1)$. Note that $P_{x,\eta}^{-1} = P_{\psi_x^\eta, \psi_x^{-1}(x)} = P_{-\Upsilon_1(x,\eta)} = P_{\Upsilon_{-1}(x,-\eta)}$. The $P_{x,\xi}$ are the parallel transport maps along geodesics on the tangent bundle. These maps are related to the Υ_t diffeomorphisms, since a computation shows that for any $(x, \eta) \in TM$ and $t \in \mathbb{R}$, $P_{x,t\eta}(\eta) = \Upsilon_{t,T}(x, \eta)$.

If (z, \mathbf{b}) is a frame, we define $\Phi_{\lambda,z,\mathbf{b}} := n_{z,M^2}^{\mathbf{b}} \circ \Phi_\lambda \circ (n_{z,T}^{\mathbf{b}})^{-1}$ and we denote $J_{\lambda,z,\mathbf{b}}$ its Jacobian. We also define $\Upsilon_{t,z,\mathbf{b}} = n_{z,T}^{\mathbf{b}} \circ \Upsilon_t \circ (n_{z,T}^{\mathbf{b}})^{-1}$ and the smooth maps from \mathbb{R}^{2n} to \mathbb{R}^n :

$$\begin{aligned} \psi_z^{\mathbf{b}} : (x, \zeta) &\mapsto n_z^{\mathbf{b}} \circ \psi \circ (n_{z,T}^{\mathbf{b}})^{-1}(x, \zeta), \\ \overline{\psi_z^{\mathbf{b}}} : (x, y) &\mapsto M_{z,(n_z^{\mathbf{b}})^{-1}(x)}^{\mathbf{b}} \circ \psi_{(n_z^{\mathbf{b}})^{-1}(x)}^{-1} \circ (n_z^{\mathbf{b}})^{-1}(y). \end{aligned}$$

Noting $\psi_{z,x}^{\mathbf{b}}(\zeta) := \psi_z^{\mathbf{b}}(x, \zeta)$ and $\overline{\psi_{z,x}^{\mathbf{b}}}(y) := \overline{\psi_z^{\mathbf{b}}}(x, y)$, we have $(\psi_{z,x}^{\mathbf{b}})^{-1} = \overline{\psi_{z,x}^{\mathbf{b}}}$. A computation shows that for any $(x, \zeta, y) \in \mathbb{R}^{3n}$,

$$\Phi_{\lambda,z,\mathbf{b}}(x, \zeta) = (\psi_z^{\mathbf{b}}(x, \lambda\zeta), \psi_z^{\mathbf{b}}(x, -(1-\lambda)\zeta)), \quad \Phi_{\lambda,z,\mathbf{b}}^{-1}(x, y) = (m_{\lambda,z,\mathbf{b}}(x, y), \xi_{\lambda,z,\mathbf{b}}(x, y)) \quad (5.16)$$

where we defined the following functions: $m_{\lambda,z,\mathbf{b}}(x, y) := \psi_z^{\mathbf{b}}(x, \lambda \overline{\psi_z^{\mathbf{b}}}(x, y))$, $\xi_{0,z,\mathbf{b}} := -\overline{\psi_z^{\mathbf{b}}}$ and for $\lambda \neq 0$, $\xi_{\lambda,z,\mathbf{b}}(x, y) := \frac{1}{\lambda} \overline{\psi_z^{\mathbf{b}}}(m_{\lambda,z,\mathbf{b}}(x, y), x)$. We also obtain for $t \in [-1, 1]$, $(x, \zeta) \in \mathbb{R}^{2n}$,

$$\Upsilon_{t,z,\mathbf{b}}(x, \zeta) = (\psi_z^{\mathbf{b}}(x, t\zeta), \frac{-1}{t} \overline{\psi_z^{\mathbf{b}}}(\psi_z^{\mathbf{b}}(x, t\zeta), x)) =: (\psi_z^{\mathbf{b}}(x, t\zeta), \Upsilon_{t,T}^{z,\mathbf{b}}(x, \zeta)), \quad (5.17)$$

and $\Upsilon_{0,z,\mathbf{b}} = \text{Id}_{\mathbb{R}^{2n}}$. Note that $\Upsilon_{t,z,\mathbf{b}}(x, 0) = (x, 0)$ for any $x \in \mathbb{R}^n$ and $\Upsilon_{t,T}^{z,\mathbf{b}} = \frac{1}{t} \Upsilon_{1,T}^{z,\mathbf{b}} \circ I_{1,t}$ where $I_{r,r'}$ is the diagonal matrix with coefficients $I_{ii} = r$ for $i \leq n$ for $1 \leq i \leq n$ and $I_{ii} = r'$ for $n+1 \leq i \leq 2n$.

5.3.2 \mathcal{O}_M -linearizations

We intent to use the linearization to define topological isomorphisms between rapidly decaying section on TM and $M \times M$. We thus need a control at infinity over the derivatives of the linearization ψ .

We note $\tau^{z,\mathbf{b}} = \tau^z \circ (n_{z,M^2}^{\mathbf{b}})^{-1} \in C^\infty(\mathbb{R}^{2n}, L(E_z))$. Remark that for any $(x, y) \in \mathbb{R}^{2n}$, $\tau^{z,\mathbf{b}}(x, y)$ is an unitary operator on E_z . We will also need the following functions parametrized by $t \in \mathbb{R}$: $\tau_t(x, \eta) := \tau_x(\psi_x(t\eta))$ for any $(x, \eta) \in TM$ and $\tau_t^{z,\mathbf{b}}(x, \zeta) := \tau^{z,\mathbf{b}}(x, \psi_z^{\mathbf{b}}(x, t\zeta))$.

Definition 5.3.4. A linearization ψ on the exponential manifold $(M, \exp, E, d\mu)$ is said to be a \mathcal{O}_M -linearization if for any frame (z, \mathbf{b}) the functions $\psi_z^{\mathbf{b}}$ and $\overline{\psi_z^{\mathbf{b}}}$ are in $\mathcal{O}_M(\mathbb{R}^{2n}, \mathbb{R}^n)$ and the functions $\tau_1^{z,\mathbf{b}}$ and $(\tau_1^{z,\mathbf{b}})^{-1}$ are in $\mathcal{O}_M(\mathbb{R}^{2n}, L(E_z))$. We will say that $(M, \exp, E, d\mu, \psi)$ has a \mathcal{O}_M -bounded geometry, if it the case of $(M, \exp, E, d\mu)$ and ψ is a \mathcal{O}_M -linearization.

Lemma 5.3.5. *Suppose that ψ is a \mathcal{O}_M -linearization. Then for any frame (z, \mathbf{b}) , $\lambda \in [0, 1]$ and $t \in [-1, 1]$,*

- (i) $\Phi_{\lambda, z, \mathbf{b}} \in \mathcal{O}_M^\times(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ and $J_{\lambda, z, \mathbf{b}} \in \mathcal{O}_M^\times(\mathbb{R}^{2n})$,
- (ii) $\Upsilon_{t, z, \mathbf{b}} \in \mathcal{O}_M^\times(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ and $J(\Upsilon_{t, z, \mathbf{b}}) \in \mathcal{O}_M^\times(\mathbb{R}^{2n})$,
- (iii) $\tau_t^{z, \mathbf{b}}$ and $(\tau_t^{z, \mathbf{b}})^{-1}$ are in $\mathcal{O}_M(\mathbb{R}^{2n}, L(E_z))$.

Proof. (i) By (5.16), we have $\Phi_{\lambda, z, \mathbf{b}} = (\psi_z^{\mathbf{b}} \circ I_{1, \lambda}, \psi_z^{\mathbf{b}} \circ I_{1, \lambda-1})$ and $\Phi_{\lambda, z, \mathbf{b}}^{-1} = (m_{\lambda, z, \mathbf{b}}, \xi_{\lambda, z, \mathbf{b}})$ where $m_{\lambda, z, \mathbf{b}} = \psi_z^{\mathbf{b}} \circ I_{1, \lambda} \circ (\pi_1, \overline{\psi_z^{\mathbf{b}}})$ and if $\lambda \neq 0$, $\xi_\lambda = \frac{1}{\lambda} \overline{\psi_z^{\mathbf{b}}} \circ (m_{\lambda, z, \mathbf{b}}, \pi_1)$, while $\xi_{0, z, \mathbf{b}} = -\overline{\psi_z^{\mathbf{b}}}$. Thus, the result is a consequence Lemma 5.2.14 (iii) and (vi).

(ii) By (5.17), we have for $t \neq 0$, $\Upsilon_{t, z, \mathbf{b}} = (\psi_z^{\mathbf{b}} \circ I_{1, t}, \frac{-1}{t} \overline{\psi_z^{\mathbf{b}}} \circ (\psi_z^{\mathbf{b}} \circ I_{1, t}, \pi_1))$. The result follows again from Lemma 5.2.14 (iii) and (vi).

(iii) We have $\tau_t^{z, \mathbf{b}} = \tau_1^{z, \mathbf{b}} \circ I_{1, t}$ and $(\tau_t^{z, \mathbf{b}})^{-1} = (\tau_1^{z, \mathbf{b}})^{-1} \circ I_{1, t}$ so the result follows from Lemma 5.2.14 (iii). \square

The following lemma shows that we can obtain topological isomorphisms on spaces of rapidly decaying functions from the functions τ_t and Φ_λ .

Lemma 5.3.6. *Let $p \in \mathbb{N}^*$, $\tau \in \mathcal{O}_M^\times(\mathbb{R}^p, GL(E_z))$ and $\Phi \in \mathcal{O}_M^\times(\mathbb{R}^p, \mathbb{R}^p)$. Then the maps $L_\tau := u \mapsto \tau u$, $R_\tau := u \mapsto u\tau$ and $C_\Phi := u \mapsto u \circ \Phi$ are topological isomorphisms of $\mathcal{S}(\mathbb{R}^p, L(E_z))$.*

Proof. Since $L_\tau^{-1} = L_{\tau^{-1}}$, $R_\tau^{-1} = R_{\tau^{-1}}$ and $C_\Phi^{-1} = C_{\Phi^{-1}}$, we only need to check the continuity of L_τ , R_τ and C_Φ . The continuity of L_τ and R_τ is a direct application of Leibniz formula. Let ν be a p -multi-index and $r \in \mathbb{N}$. Theorem 5.2.11 implies that for any $u \in \mathcal{S}(\mathbb{R}^p, L(E_z))$,

$$q_{\nu, N}(u \circ \Phi) \leq \sum_{|\lambda| \leq |\nu|} \sup_{x \in \mathbb{R}^p} \langle x \rangle^N |P_{\nu, \lambda}(\Phi)(x)| \left\| (\partial^\lambda u) \circ \Phi(x) \right\|_{L(E_z)}$$

where the functions $P_{\nu, \lambda}(\Phi)$ are such that $|P_{\nu, \lambda}(\Phi)(x)| \leq C_\nu \langle x \rangle^{q_\nu}$ for a $q_\nu \in \mathbb{N}^*$ and a $C_\nu > 0$. Since $\langle \Phi^{-1}(x) \rangle \leq C \langle x \rangle^r$ for a $r \in \mathbb{N}^*$ and a $C > 0$, we see that there is $C'_\nu > 0$ such that $q_{\nu, N}(u \circ \Phi) \leq C'_\nu \sum_{|\lambda| \leq |\nu|} q_{\lambda, (q_\nu + N)r}(u)$, which gives the result. \square

Lemma 5.3.7. *If $(M, \exp, E, d\mu)$ has a \mathcal{O}_M -bounded geometry and ψ is a linearization such that there exists (z_0, \mathbf{b}_0) such that the functions $\psi_{z_0}^{\mathbf{b}_0}$, $\overline{\psi_{z_0}^{\mathbf{b}_0}}$ are in $\mathcal{O}_M(\mathbb{R}^{2n}, \mathbb{R}^n)$ and $\tau_1^{z_0, \mathbf{b}_0}$, $(\tau_1^{z_0, \mathbf{b}_0})^{-1}$ are in $\mathcal{O}_M(\mathbb{R}^{2n}, L(E_{z_0}))$, then ψ is a \mathcal{O}_M -linearization.*

Proof. The result is a direct consequence of the formulas $\psi_z^{\mathbf{b}} = \psi_{z, z_0}^{\mathbf{b}, \mathbf{b}_0} \circ \psi_{z_0}^{\mathbf{b}_0} \circ \psi_{z_0, z, T}^{\mathbf{b}_0, \mathbf{b}}$, $\overline{\psi_z^{\mathbf{b}}}(y) = (d\psi_{z_0, z}^{\mathbf{b}_0, \mathbf{b}})^{-1} \overline{\psi_{z_0}^{\mathbf{b}_0}} \circ \psi_{z_0, z, M^2}^{\mathbf{b}_0, \mathbf{b}}(x, y)$ and $\tau_z^{z, \mathbf{b}} = (\tau_z^{-1} \tau_{z_0}) \circ \pi_2 \circ (n_{z, M^2}^{\mathbf{b}})^{-1} \tau_{z_0, \mathbf{b}_0} \circ \psi_{z_0, z, M^2}^{\mathbf{b}_0, \mathbf{b}} (\tau_{z_0}^{-1} \tau_z) \circ \pi_1 \circ (n_{z, M^2}^{\mathbf{b}})^{-1}$. \square

5.3.3 Symbol maps and λ -quantization

Assumption 5.3.8. *We suppose in this section and in section 5.3.4 that $(M, \exp, E, d\mu, \psi)$ has a \mathcal{O}_M -bounded geometry.*

The operator \mathcal{F} is a topological isomorphism from $\mathcal{S}'(TM, L(E))$ onto $\mathcal{S}'(T^*M, L(E))$. We shall now introduce a topological isomorphism between $\mathcal{S}'(M \times M, L(E))$ and $\mathcal{S}'(TM, L(E))$. We define the linear application Γ_λ from $C^\infty(M \times M, L(E))$ into $C^\infty(TM, L(E))$:

$$\Gamma_\lambda(K) : v \mapsto K^{\pi(v)} \circ \Phi_\lambda(v).$$

As a consequence, $\Gamma_\lambda(K) = \tau_\lambda^{-1}(K \circ \Phi_\lambda) \tau_{\lambda-1}$ and $\Gamma_\lambda^{-1}(a) = (\tau_\lambda a \tau_{\lambda-1}^{-1}) \circ \Phi_\lambda^{-1}$. For a given frame (z, \mathbf{b}) , we denote $\Gamma_{\lambda,z,\mathbf{b}} := T_{z,\mathbf{b},T} \circ \Gamma_\lambda \circ T_{z,\mathbf{b},M^2}^{-1}$. A computation shows that for any smooth function $u \in C^\infty(\mathbb{R}^{2n}, L(E_z))$, $\Gamma_{\lambda,z,\mathbf{b}}(u) = (\tau_\lambda^{z,\mathbf{b}})^{-1}(u \circ \Phi_{\lambda,z,\mathbf{b}}) \tau_{\lambda-1}^{z,\mathbf{b}}$.

Let us define the smooth strictly positive functions on \mathbb{R}^{2n} and $M \times M$ respectively:

$$\mu_{\lambda,z,\mathbf{b}}(x, y) := \frac{\mu_{z,\mathbf{b}}(x) \mu_{z,\mathbf{b}}(y)}{\mu_{z,\mathbf{b}}^2(m_{\lambda,z,\mathbf{b}}(x, y))} |J_{\lambda,z,\mathbf{b}}| \circ \Phi_{\lambda,z,\mathbf{b}}^{-1}(x, y) \quad \mu_\lambda := \mu_{\lambda,z,\mathbf{b}} \circ (n_z^{\mathbf{b}}, n_z^{\mathbf{b}}). \quad (5.18)$$

It is straightforward to check that μ_λ is indeed independent of (z, \mathbf{b}) . Note that $\mu_{1-\lambda}(x, y) = \mu_\lambda(y, x)$. Since $\mu_{\lambda,z,\mathbf{b}} \in \mathcal{O}_M^\times(\mathbb{R}^{2n})$, the operator of multiplication M_{μ_λ} is a topological isomorphism on $\mathcal{S}(M \times M, L(E))$. Note also that $\Gamma_\lambda \circ M_{\mu_\lambda} = M_{\mu_\lambda \circ \Phi_\lambda} \circ \Gamma_\lambda$.

Proposition 5.3.9. Γ_λ is a topological isomorphism from $\mathcal{S}(M \times M, L(E))$ onto $\mathcal{S}(TM, L(E))$. Moreover, $\tilde{\Gamma}_\lambda \circ j_{M^2} = j_{TM} \circ \Gamma_\lambda \circ M_{\mu_\lambda}$, where $\tilde{\Gamma}_\lambda := \Gamma_\lambda^{-1*}$.

Proof. Let (z, \mathbf{b}) be a frame. It suffices to prove that $\Gamma_{\lambda,z,\mathbf{b}}$ is a topological isomorphism from $\mathcal{S}(\mathbb{R}^{2n}, L(E_z))$ onto itself. Since $\Gamma_{\lambda,z,\mathbf{b}} = L_{(\tau_\lambda^{z,\mathbf{b}})^{-1}} \circ R_{\tau_{\lambda-1}^{z,\mathbf{b}}} \circ C_{\Phi_{\lambda,z,\mathbf{b}}}$, the result follows from Lemma 5.3.6 and Lemma 5.3.5 (i) and (iii). Let $u, v \in \mathcal{S}(\mathbb{R}^{2n}, L(E_z))$. We have (with j the canonical inclusion from $\mathcal{S}(\mathbb{R}^{2n}, L(E_z))$ into $\mathcal{S}'(\mathbb{R}^{2n}, L(E_z))$):

$$\begin{aligned} (\tilde{\Gamma}_{\lambda,z,\mathbf{b}} \circ j(u))(v) &= \int_{\mathbb{R}^{2n}} \text{Tr} (u(x, y) (\Gamma_{\lambda,z,\mathbf{b}}^{-1}(v)(x, y))^*) dx dy \\ &= \int_{\mathbb{R}^{2n}} \text{Tr} ((\tau_\lambda^{z,\mathbf{b}})^{-1} \circ \Phi_{\lambda,z,\mathbf{b}}^{-1}(x, y) u(x, y) \tau_{\lambda-1}^{z,\mathbf{b}} \circ \Phi_{\lambda,z,\mathbf{b}}^{-1}(x, y) \\ &\quad v^* \circ \Phi_{\lambda,z,\mathbf{b}}^{-1}(x, y)) dx dy \\ &= \int_{\mathbb{R}^{2n}} \text{Tr} (\Gamma_{\lambda,z,\mathbf{b}}(u)(m, \zeta) v^*(m, \zeta)) |J_{\lambda,z,\mathbf{b}}|(m, \zeta) dm d\zeta \\ &= (j \circ M_{|J_{\lambda,z,\mathbf{b}}|} \circ \Gamma_{\lambda,z,\mathbf{b}}(u))(v) \end{aligned}$$

where we used the following change of variables $(m, \zeta) := \Phi_{\lambda,z,\mathbf{b}}^{-1}(x, y)$. Thus, we have $\tilde{\Gamma}_{\lambda,z,\mathbf{b}} \circ j = j \circ M_{|J_{\lambda,z,\mathbf{b}}|} \circ \Gamma_{\lambda,z,\mathbf{b}}$. The relation $\tilde{\Gamma}_\lambda \circ j_{M^2} = j_{TM} \circ \Gamma_\lambda \circ M_{\mu_\lambda}$ now follows since $M_{|J_{\lambda,z,\mathbf{b}}|} \circ \Gamma_{\lambda,z,\mathbf{b}} = \Gamma_{\lambda,z,\mathbf{b}} \circ M_{|J_{\lambda,z,\mathbf{b}}| \circ \Phi_{\lambda,z,\mathbf{b}}^{-1}}$, $T_{z,\mathbf{b},T}^* \circ j \circ M_{\mu_{z,\mathbf{b}}}^2 = j_{TM} \circ T_{z,\mathbf{b},T}^{-1}$ and $T_{z,\mathbf{b},M^2}^* \circ j \circ M_{\mu_{z,\mathbf{b}} \otimes \mu_{z,\mathbf{b}}} = j_{M^2} \circ T_{z,\mathbf{b},M^2}^{-1}$. \square

As a consequence, $\tilde{\Gamma}_\lambda$ is a topological isomorphism from the space of tempered distributional $L(E)$ -sections on $M \times M$, $\mathcal{S}'(M \times M, L(E))$ onto $\mathcal{S}'(TM, L(E))$ and when restricted (in the sense of the previous continuous inclusions) to $\mathcal{S}(M \times M, L(E))$, is equal to $\Gamma_\lambda \circ M_{\mu_\lambda}^{-1}$, so provides a topological isomorphism from $\mathcal{S}(M \times M, L(E))$ onto $\mathcal{S}(TM, L(E))$. Fourier transform coupled with $\tilde{\Gamma}_\lambda$ lead us to the following natural isomorphism from $\mathcal{S}'(M \times M, L(E))$ onto $\mathcal{S}'(T^*M, L(E))$.

Definition 5.3.10. Let $\lambda \in [0, 1]$. The λ -symbol map is the topological isomorphism from $\mathcal{S}'(M \times M, L(E))$ onto $\mathcal{S}'(T^*M, L(E))$: $\sigma_\lambda := \mathcal{F} \circ \tilde{\Gamma}_\lambda$. The λ -quantization map is the inverse of σ_λ , noted \mathfrak{Op}_λ .

Thus, the data of a tempered distributional section on the cotangent bundle (i.e. an element of $\mathcal{S}'(T^*M, L(E))$) determines in a unique way (for a given λ), an operator continuous from \mathcal{S} to \mathcal{S}' , and vice versa. Remark that $\sigma_\lambda \circ j_{M^2} = j_{T^*M} \circ \mathcal{F} \circ \Gamma_\lambda \circ M_{\mu_\lambda}$ and $\mathfrak{Op}_\lambda \circ j_{T^*M} =$

$j_{M^2} \circ M_{1/\mu_\lambda} \circ \Gamma_\lambda^{-1} \circ \overline{\mathcal{F}}$. If (z, \mathbf{b}) is a frame then, noting $\mathfrak{Op}_{\lambda,z,\mathbf{b}} := \widetilde{T}_{z,\mathbf{b},M^2} \circ \mathfrak{Op}_\lambda \circ \widetilde{T}_{z,\mathbf{b},*}^{-1}$, we obtain $\mathfrak{Op}_{\lambda,z,\mathbf{b}} = \Gamma_{\lambda,z,\mathbf{b}}^* \circ M_{\mu_{z,\mathbf{b}}}^* \circ \mathcal{F}_P^*$ so that for any $u \in \mathcal{S}(\mathbb{R}^{2n}, L(E_z))$ and $b \in \mathcal{O}_M(\mathbb{R}^{2n}, L(E_z))$,

$$\langle \mathfrak{Op}_{\lambda,z,\mathbf{b}}(b), u \rangle = \int_{\mathbb{R}^{3n}} e^{2\pi i \langle \zeta, \vartheta \rangle} \text{Tr}(\mu b(x, \vartheta) (\Gamma_{\lambda,z,\mathbf{b}}(u))^*(x, \zeta)) d\zeta d\vartheta dx. \quad (5.19)$$

where $\mu b : (x, \vartheta) \mapsto \mu_{z,\mathbf{b}}(x) b(x, \vartheta)$.

5.3.4 Moyal product

The applications $\mathfrak{Op}_0, \mathfrak{Op}_1, \mathfrak{Op}_W := \mathfrak{Op}_{\frac{1}{2}}$ are respectively the normal, antinormal and Weyl quantization maps. Remark that for any $T \in \mathcal{S}'(T^*M, L(E))$, $\mathfrak{Op}_\lambda(T^*) = (\mathfrak{Op}_{1-\lambda}(T))^\dagger$. In particular

$$\mathfrak{Op}_0(T^*) = (\mathfrak{Op}_1(T))^\dagger, \quad \mathfrak{Op}_W(T^*) = (\mathfrak{Op}_W(T))^\dagger$$

where \dagger is the topological isomorphism of $\mathcal{S}'(M \times M, L(E))$ defined as $\langle K^\dagger, u \rangle := \overline{\langle K, u^* \circ j \rangle}$ with j the diffeomorphism on $M \times M : (x, y) \mapsto (y, x)$ and $u \in \mathcal{S}(M \times M, L(E))$. The kernel of the adjoint A^\dagger of any operator $A \in L(\mathcal{S}, \mathcal{S}')$ is $(K_A)^\dagger$. As a consequence, σ_λ is a linear topological isomorphism (and a $*$ -isomorphism in the case of the Weyl quantization) from the algebra $\mathfrak{R}(\mathcal{S}) = L(\mathcal{S}, \mathcal{S}) \cap L(\mathcal{S}', \mathcal{S}')$ of regular operators onto its image $\mathfrak{M}_\lambda := \sigma_\lambda(\mathfrak{R}(\mathcal{S}))$. We can transport the operator composition in the world of functions, by defining the λ -product on \mathfrak{M}_λ as

$$T \circ_\lambda T' := \sigma_\lambda(\mathfrak{Op}_\lambda(T) \mathfrak{Op}_\lambda(T'))$$

so that \mathfrak{M}_λ forms an algebra, and $\mathfrak{M}_\lambda^* = \mathfrak{M}_{1-\lambda}$. In the case of $\lambda = \frac{1}{2}$, we recover the Moyal $*$ -algebra \mathfrak{M}_W and the Moyal product \circ_W . The space $\Psi^{-\infty}(M) \simeq \mathcal{S}(M \times M, L(E))$ of isotropic smoothing operators being an $*$ -ideal of $\mathfrak{R}(\mathcal{S})$, the space $\mathcal{S}(T^*M, L(E)) = \sigma_\lambda(\Psi^{-\infty}(M))$ forms an ideal of \mathfrak{M}_λ . Since we will focus on the pseudodifferential calculus over M , we shall not investigate in this chapter the full analysis of the Moyal product over T^*M . Note however the following property on $\mathcal{S}(T^*M) := \mathcal{S}(T^*M, L(M \times \mathbb{C})) \simeq \mathcal{S}(T^*M, \mathbb{C})$:

Proposition 5.3.11. *$(\mathcal{S}(T^*M), \circ_\lambda)$ is a (noncommutative, nonunital) Fréchet algebra. Moreover,*

$$a \circ_\lambda b(x, \eta) = \int_{T_x(M) \times M} d\mu_x(\xi) d\mu(y) \int_{V_{x,\xi}^\lambda} d\mu_{x,\xi,y}^*(\theta, \theta') g_{x,\xi,y}^\lambda e^{2\pi i \omega_{x,\xi,y}^\lambda(\eta, \theta, \theta')} a(y_{x,\xi}^\lambda, \theta) b(y_{x,-\xi}^{1-\lambda}, \theta')$$

where $y_{x,\xi}^\lambda := m_\lambda(\psi_x^{\lambda\xi}, z)$, $\overline{y}_{x,\xi}^\lambda := \xi_\lambda(\psi_x^{\lambda\xi}, z)$ and

$$\begin{aligned} V_{x,\xi,y}^\lambda &:= T_{y_{x,\xi}^\lambda}^*(M) \times T_{\overline{y}_{x,-\xi}^{1-\lambda}}^*(M), & d\mu_{x,\xi,y}^*(\theta, \theta') &:= d\mu_{y_{x,\xi}^\lambda}^*(\theta) d\mu_{\overline{y}_{x,-\xi}^{1-\lambda}}^*(\theta'), \\ g_{x,\xi,y}^\lambda &:= \frac{\mu_\lambda(\psi_x^{\lambda\xi}, \psi_x^{(\lambda-1)\xi})}{\mu_\lambda(\psi_x^{\lambda\xi}, y) \mu_\lambda(y, \psi_x^{(\lambda-1)\xi})}, \\ \omega_{x,\xi,y}^\lambda(\eta, \theta, \theta') &:= \langle \theta, \overline{y}_{x,\xi}^\lambda \rangle - \langle \theta', \overline{y}_{x,-\xi}^{1-\lambda} \rangle - \langle \eta, \xi \rangle. \end{aligned}$$

Proof. The product $a \circ_\lambda b$ on $\mathcal{S}(T^*M)$ is obtained by computation of $\mathcal{F} \circ \Gamma_\lambda \circ M_{\mu_\lambda} \circ ((M_{\mu_\lambda}^{-1} \circ \Gamma_\lambda^{-1} \circ \overline{\mathcal{F}}(a)) \circ_V (M_{\mu_\lambda}^{-1} \circ \Gamma_\lambda^{-1} \circ \overline{\mathcal{F}}(b)))$, where \circ_V is the Volterra product of kernels. Since σ_λ is a topological isomorphism between $\mathcal{S}(M^2)$ and $\mathcal{S}(T^*M)$, the continuity of the Moyal product is

equivalent to the continuity of \circ_V , which is equivalent to the continuity of the following product on $\mathcal{S}(\mathbb{R}^{2n})$:

$$K \cdot K'(x, y) := \int_{\mathbb{R}^n} K(x, t) K'(t, y) \mu_{z, \mathbf{b}}(t) dt.$$

The continuity of this product is obtained by the following estimates

$$q_{p,(\alpha,\beta)}(K \cdot K') \leq C q_{2(p+r),(\alpha,0)}(K) q_{p,(0,\beta)}(K'), \quad q_{p,\nu}(K) := \sup_{(x,y) \in \mathbb{R}^{2n}} \langle (x, y) \rangle^p |\partial^\nu K(x, y)|$$

where $|\mu_{z, \mathbf{b}}(t)| \leq C_1 \langle t \rangle^{r-n-1}$ and $C := C_1 \int_{\mathbb{R}^n} \langle t \rangle^{-(n+1)} dt$. \square

Remark 5.3.12. $(\mathcal{S}(T^*M), \circ_W)$ is a $*$ -algebra since $(a \circ_W b)^* = b^* \circ_W a^*$ for any $a, b \in \mathcal{S}(T^*M)$. We can also construct another $*$ -algebra on $\mathcal{S}(T^*M)$ with the product $a \star b := \frac{1}{2}(a \circ_0 b + a \circ_1 b)$. This proves that when (H_ψ) (see Assumption 5.3.1) is not satisfied (so that no middle point exist in the classical world) we can still have a canonical star-product on $\mathcal{S}(T^*M)$ which satisfies $(a \star b)^* = b^* \star a^*$.

5.4 Symbol calculus of pseudodifferential operators

5.4.1 Symbols

Assumption 5.4.1. Let $\sigma \in [0, 1]$. We suppose in this section that (M, \exp, E) has a S_σ -bounded geometry.

The algebra $\mathfrak{R}(\mathcal{S})$ and $\Psi^{-\infty}$ are respectively too big and too small to develop a satisfactory pseudodifferential calculus that allows an efficient utilization of symbol maps. We shall in this section define some spaces of symbols that will be used to define later special algebras of pseudodifferential operators that lie between $\mathfrak{R}(\mathcal{S})$ and $\Psi^{-\infty}$.

Definition 5.4.2. A symbol of degree $(l, m) \in \mathbb{R}^2$ of type σ , on M is a smooth section $a \in C^\infty(T^*M, L(E))$ such that for any (z, \mathbf{b}) and any n -multi-indices α, β , there exists $K > 0$ such that

$$\left\| \partial_{z, \mathbf{b}}^{(\alpha, \beta)} a^z(x, \theta) \right\|_{L(E_z)} \leq K \langle x \rangle_{z, \mathbf{b}}^{\sigma(l-|\alpha|)} \langle \theta \rangle_{z, \mathbf{b}, x}^{m-|\beta|} \quad (5.20)$$

is valid for all $(x, \theta) \in T^*M$. The space of symbols of degree (l, m) and type σ is noted $S_\sigma^{l, m}$.

Remark that $S_0^{l, m}$ is independent of l , so we denote this space S_0^m . We note $S_\sigma^{-\infty} := \cap_{l, m} S_\sigma^{l, m}$ and in the case $\sigma > 0$, we define $S^{-\infty} := S_\sigma^{-\infty} = \mathcal{S}(T^*M, L(E))$ (it is independent of $\sigma > 0$). We set $S_\sigma^\infty := \cup_{l, m} S_\sigma^{l, m}$. We define similarly $S_{\sigma, z}^{l, m} := S_\sigma^{l, m}(\mathbb{R}^{2n}, L(E_z))$, without reference to a frame.

Since M has a S_σ -bounded geometry, we get the following coordinate independence of the previous definition:

Proposition 5.4.3. Let $a \in C^\infty(T^*M, L(E))$. Then $a \in S_\sigma^{l, m}$ if and only if there exists a frame (z, \mathbf{b}) such that a satisfies (5.20).

Proof. Suppose that (5.20) is satisfied for (z', \mathbf{b}') and let (z, \mathbf{b}) be another frame. For $(x, \theta) \in T^*M$ and α, β two n -multi-indices with $\nu = (\alpha, \beta) \neq 0$, we get from Equation (5.15) and Lemma 5.2.13,

$$\begin{aligned} \|\partial_{z, \mathbf{b}}^\nu a^z(x, \theta)\|_{L(E_z)} &\leq K \sum_{\alpha', \beta'} \sum_{\rho \leq \lambda \leq \nu'} \langle x \rangle_{z, \mathbf{b}}^{\sigma(|\alpha'| - |\alpha|)} \langle \theta \rangle_{z, \mathbf{b}, x}^{|\beta'| - |\beta|} \langle x \rangle_{z', \mathbf{b}'}^{\sigma(|\lambda^1| - |\alpha'|)} \\ &\quad \times \langle x \rangle_{z', \mathbf{b}'}^{\sigma(l - |\rho^1|)} \langle \theta \rangle_{z', \mathbf{b}', x}^{m - |\beta'|} \langle x \rangle_{z', \mathbf{b}'}^{\sigma(|\rho^1| - |\lambda^1|)}. \end{aligned}$$

Using (5.2), (5.3) and the fact that $|\alpha| \geq |\rho^1|$, we get the result. \square

Corollary 5.4.4. *If $a \in C^\infty(T^*M, L(E))$, then $a \in S_\sigma^{l, m}$ if and only if for any (z, \mathbf{b}) , $a^z \circ (n_{z, *}^{\mathbf{b}})^{-1} \in S_\sigma^{l, m}(\mathbb{R}^{2n}, L(E_z))$, or equivalently, there exists (z, \mathbf{b}) such that $a^z \circ (n_{z, *}^{\mathbf{b}})^{-1} \in S_\sigma^{l, m}(\mathbb{R}^{2n}, L(E_z))$.*

We see that $S_\sigma^{l, m} \cdot S_\sigma^{l', m'} \subseteq S_\sigma^{l+l', m+m'}$ where \cdot is the composition of sections induced by the matrix product on the fibers of $L(E)$. Moreover, $S_\sigma^{l, m} \subseteq S_\sigma^{l', m'}$ for $m \leq m'$ and $l \leq l'$. Thus, S_σ^∞ is a $*$ -algebra, which is bigraduated for $\sigma > 0$ and graduated for $\sigma = 0$. Remark also that $S^{-\infty} \cdot S_0^m$ and $S_0^m \cdot S^{-\infty}$ are included in $S^{-\infty}$. Note that if $f \in S_\sigma^{l, m}(T^*M)$ (a symbol where M has its trivial bundle $M \times \mathbb{C}$), then $a_f(x, \theta) := f(x, \theta)I_{L(E_x)}$ defines a symbol in $S_\sigma^{l, m}$. Such symbols will be called scalar symbols. Note also that if $a \in S_\sigma^{l, m}$, then $\partial_{z, \mathbf{b}}^{(\alpha, \beta)} a := (\tau_z \circ \pi)(\partial_{z, \mathbf{b}}^{(\alpha, \beta)} a^z)(\tau_z^{-1} \circ \pi) \in S_\sigma^{l-|\alpha|, m-|\beta|}$.

If $f \in S_\sigma(\mathbb{R}^n)$ then $(x, \vartheta) \mapsto f(x) \text{Id}_{L(E_z)} \in S_\sigma^{0, 0}(\mathbb{R}^n, L(E_z))$. In particular $(x, \vartheta) \mapsto \mu_{z, \mathbf{b}}^{\pm 1}(x) \text{Id}_{L(E_z)} \in S_\sigma^{0, 0}(\mathbb{R}^n, L(E_z))$ if $d\mu$ is a S_σ^\times -density.

Remark 5.4.5. *We note $PS_\sigma^{l, m}(\mathbb{R}^{2n}, L(E_z))$ the subspace of $S_\sigma^{l, m}(\mathbb{R}^{2n}, L(E_z))$ consisting of functions of the form $\sum_{1 \leq i \leq (\dim E_z)^2} P_i e_i$ where (e_i) is a linear basis of $L(E_z)$ and P_i are of the form $\sum_\beta c_{i, \beta}(x) \vartheta^\beta$ (finite sum over the n -multi-indices β), where for any i, β , $\partial^\alpha c_{i, \beta}(x) = \mathcal{O}(\langle x \rangle^{\sigma(l-|\alpha|)})$ for any n -multi-indices α , and $m = \max_i \deg_\vartheta P_i$. We check that this definition is independent of the chosen basis (e_i) .*

*We call polynomial symbol of order l, m and type σ any section of the form $(\tau_z \circ \pi)(P \circ n_{z, *}^{\mathbf{b}})(\tau_z^{-1} \circ \pi)$ where $P \in PS_\sigma^{l, m}(\mathbb{R}^{2n}, L(E_z))$ and (z, \mathbf{b}) is a frame. This definition is independent of (z, \mathbf{b}) . We note $PS_\sigma^{l, m}$ the subspace of $S_\sigma^{l, m}$ consisting of polynomial symbols of order l, m and type σ . Remark that the section $I : (x, \theta) \mapsto I_{L(E_x)}$ is in $PS_1^{0, 0}$.*

We now topologize the symbol spaces:

Lemma 5.4.6. *The following semi-norms on $S_\sigma^{l, m}$, for $N \in \mathbb{N}$,*

$$q_{(\alpha, \beta)}(a) := \sup_{(x, \theta) \in T^*M} \langle x \rangle_{z, \mathbf{b}}^{\sigma(|\alpha| - l)} \langle \theta \rangle_{z, \mathbf{b}, x}^{|\beta| - m} \left\| \partial_{z, \mathbf{b}}^{(\alpha, \beta)} a^z(x, \theta) \right\|_{L(E_z)}$$

*determine a Fréchet topology on $S_\sigma^{l, m}$, which is independent of (z, \mathbf{b}) . The applications $T_{z, \mathbf{b}, *}$ are topological isomorphisms from $S_\sigma^{l, m}$ onto $S_\sigma^{l, m}(\mathbb{R}^{2n}, L(E_z))$. The following inclusions are continous for these topologies: $S_\sigma^{l, m} \cdot S_\sigma^{l', m'} \subseteq S_\sigma^{l+l', m+m'}$, $S_\sigma^{l, m} \subseteq S_\sigma^{l', m'}$ ($m \leq m'$ and $l \leq l'$) and $S_\sigma^{-\infty} \subseteq S_\sigma^{l, m}$. Moreover, the last inclusion is dense when $S_\sigma^{l, m}$ has the topology of $S_\sigma^{l', m'}$ for $m < m'$ and $l < l'$.*

Proof. The independence of the topology for (z, \mathbf{b}) follows from the easily checked estimate for any (α, β) ,

$$q_{(\alpha, \beta)}^{(z, \mathbf{b})}(a) \leq K_{\alpha, \beta} \sum_{\substack{0 \leq |(\alpha', \beta')| \leq |(\alpha, \beta)| \\ |\beta'| \geq |\beta|, \gamma \leq \alpha'}} q_{(\gamma, \beta')}^{(z', \mathbf{b}')} (a).$$

where $K_{\alpha, \beta} > 0$. By construction the applications $T_{z, \mathbf{b}, *}$ are clearly topological isomorphisms from $S_{\sigma}^{l, m}$ onto $S_{\sigma}^{l, m}(\mathbb{R}^{2n}, L(E_z))$. The continuity of $S_{\sigma}^{l, m} \cdot S_{\sigma}^{l', m'} \subseteq S_{\sigma}^{l+l', m+m'}$, $S_{\sigma}^{l, m} \subseteq S_{\sigma}^{l', m'}$ ($m \leq m'$ and $l \leq l'$) and $S_{\sigma}^{-\infty} \subseteq S_{\sigma}^{l, m}$ are straightforward. Following [105], to prove the density result, we shall prove the stronger property: for any $a \in S_{\sigma}^{l, m}(\mathbb{R}^{2n}, L(E_z))$ the sequence

$$a_p(x, \vartheta) := (\rho(x/p))^{1-\delta_{\sigma, 0}} \rho(\vartheta/p) a(x, \vartheta)$$

converges to a for the topology of $S_{\sigma}^{l', m'}(\mathbb{R}^{2n}, L(E_z))$ where $m' > m$ and $l' > l$. Here $\rho \in C_c^{\infty}(\mathbb{R}^n, [0, 1])$ with $\rho = 1$ on $B(0, 1)$ and $\rho = 0$ on $\mathbb{R}^n \setminus B(0, 2)$. First, it is clear that $a_p \in S_{\sigma}^{-\infty}(\mathbb{R}^{2n}, L(E_z))$. Noting $R_p(x, \vartheta) := \langle x \rangle^{\sigma(|\alpha|-l')} \langle \vartheta \rangle^{|\beta|-m'} \|\partial^{(\alpha, \beta)}(a - a_p)(x, \vartheta)\|_{L(E_z)}$ for a given $2n$ -multi-index $\nu := (\alpha, \beta)$, we get with Leibniz rule, for a $K > 0$ (by convention $\nu' < \nu$ if and only if $\nu' \leq \nu$ and $\nu' \neq \nu$):

$$\frac{1}{K} R_p(x, \vartheta) \leq \Delta_p(x, \vartheta) \langle x \rangle^{\sigma(l-l')} \langle \vartheta \rangle^{m-m'} + \sum_{\nu' < \nu} |\partial^{\nu-\nu'} \Delta_p(x, \vartheta)| \langle x \rangle^{\sigma(l-l'+|\alpha|-|\alpha'|)} \langle \vartheta \rangle^{m-m'+|\beta|-|\beta'|}$$

where $\Delta_p(x, \vartheta) := 1 - (\rho(x/p))^{1-\delta_{\sigma, 0}} \rho(\vartheta/p)$. Suppose that $\sigma = 0$. In that case, $|\Delta_p(x, \vartheta)| \leq 1_{[p, +\infty]}(\vartheta)$ and if $\nu' < \nu$,

$$|\partial^{\nu-\nu'} \Delta_p(x, \vartheta)| \leq \delta_{\alpha, \alpha'} K_{\beta} p^{-|\beta|+|\beta'|} 1_{[p, 2p]}(\vartheta) \quad (5.21)$$

where $1_{[r, r']}$ is the characteristic function of the annulus $A_{r, r'} := \{\vartheta \in \mathbb{R}^n : r \leq \|\vartheta\| \leq r'\}$ and $K_{\beta} := \sup_{\beta' < \beta} \|\partial^{\beta-\beta'} \rho\|_{\infty}$. As a consequence, for $K' > 0$,

$$\frac{1}{K} R_p(x, \vartheta) \leq \langle p \rangle^{m-m'} + K_{\beta} \sum_{\nu' < \nu} \delta_{\alpha, \alpha'} 1_{[p, 2p]}(\vartheta) p^{-|\beta|+|\beta'|} \langle \vartheta \rangle^{m-m'+|\beta|-|\beta'|} \leq K' \langle p \rangle^{m-m'}$$

and the result follows. Suppose now $\sigma \neq 0$. In that case $|\Delta_p(x, \vartheta)| \leq 1_{F_p}(x, \vartheta)$ where $F_p := \mathbb{R}^{2n} - B(0, p)^2$ and if $\nu' < \nu$, for a constant $K_{\nu} > 0$

$$|\partial^{\nu-\nu'} \Delta_p(x, \vartheta)| \leq K_{\nu} 1_{[\text{sgn}(\alpha-\alpha')p, 2p]}(x) 1_{[\text{sgn}(\beta-\beta')p, 2p]}(\vartheta) p^{-|\nu|+|\nu'|}. \quad (5.22)$$

As a consequence, for $K', K'' > 0$, and with $r := \max\{m - m', \sigma(l - l')\} < 0$,

$$\frac{1}{K} R_p(x, \vartheta) \leq \langle p \rangle^r + K' \sum_{\nu' < \nu} 1_{[\text{sgn}(\alpha-\alpha')p, 2p]}(x) 1_{[\text{sgn}(\beta-\beta')p, 2p]}(\vartheta) \langle x \rangle^{\sigma(l-l')} \langle \vartheta \rangle^{m-m'} \leq K'' \langle p \rangle^r$$

and the result follows. \square

Note that $S^{-\infty} := \cap_{l, m} S_{\sigma > 0}^{-\infty} = \mathcal{S}(T^*M, L(E))$ and the equality is also valid for the topologies. The following lemma shows that the symbols of $S_{\sigma}^{l, m}$ are tempered distributional sections on T^*M .

Lemma 5.4.7. *The application j_{T^*M} is injective and continuous from $S_\sigma^{l,m}$ into $\mathcal{S}'(T^*M, L(E))$.*

Proof. Since we have the following commutative diagram

$$\begin{array}{ccccc} S_\sigma^{l,m} & \xrightarrow{j_{T^*M}} & \mathcal{S}'(T^*M, L(E)) \\ T_{z,b,*} \downarrow & & \uparrow T_{z,b,*}^* \\ S_\sigma^{l,m}(\mathbb{R}^{2n}, L(E_z)) & \xrightarrow{i} \mathcal{O}_M(\mathbb{R}^{2n}, L(E_z)) \xrightarrow{j} & \mathcal{S}'(\mathbb{R}^{2n}, L(E_z)) \end{array}$$

where $T_{z,b,*}^*$ is the adjoint of $T_{z,b,*}$ on $\mathcal{S}(T^*M, L(E))$ and $\mathcal{O}_M(\mathbb{R}^{2n}, L(E_z))$ is the locally convex complete Hausdorff space of $L(E_z)$ -valued functions on \mathbb{R}^{2n} with polynomially bounded derivatives, it is sufficient to check that the natural injection i is continuous from $S_\sigma^{l,m}(\mathbb{R}^{2n}, L(E_z))$ into $\mathcal{O}_M(\mathbb{R}^{2n}, L(E_z))$. This is obtained by the following estimate, for any $\varphi \in \mathcal{S}(\mathbb{R}^{2n})$ and $\nu = (\alpha, \beta)$ $2n$ -multi-index,

$$\sup_{(x,\vartheta) \in \mathbb{R}^{2n}} \|\varphi \partial^\nu a(x, \vartheta)\|_{L(E_z)} \leq K_{\varphi,\nu} q_\nu(a)$$

where $K_{\varphi,\nu} := \sup_{(x,\vartheta) \in \mathbb{R}^{2n}} |\varphi(x, \vartheta) \langle x \rangle^{\sigma(l-|\alpha|)} \langle \vartheta \rangle^{m-|\beta|}|$. \square

Definition 5.4.8. Let $(a_j)_{j \in \mathbb{N}^*}$ be a sequence in $S_\sigma^{l_j, m_j}$ where (l_j) and (m_j) are real strictly decreasing sequences such that $\lim_{j \rightarrow \infty} l_j = \lim_{j \rightarrow \infty} m_j = -\infty$. We say that a is an asymptotic expansion of $(a_j)_{j \in \mathbb{N}^*}$ and we denote

$$a \sim \sum_{j=1}^{\infty} a_j$$

if $a \in C^\infty(T^*M, L(E))$ is such that $a - \sum_{j=1}^{k-1} a_j \in S_\sigma^{l_k, m_k}$ for any $k \in \mathbb{N}$ with $k \geq 2$. In particular, we have $a \in S_\sigma^{l_1, m_1}$.

We need asymptotic summation of symbols modulo $S_\sigma^{-\infty}$. The following result of asymptotic completeness is based on a classical method [127] of approximation of series by weighting summands $a_j(x, \theta)$ with functions which “cut” a neighborhood of zero in the domain of x (if $\sigma \neq 0$) and θ . The idea is that the part we cut is bigger and bigger when $j \rightarrow \infty$ so that convergence occurs.

Lemma 5.4.9. *Let $(a_j)_{j \in \mathbb{N}^*}$ be a sequence in $S_\sigma^{l_j, m_j}$ where (l_j) and (m_j) are real strictly decreasing sequences such that $\lim_{j \rightarrow \infty} l_j = \lim_{j \rightarrow \infty} m_j = -\infty$. Then*

- (i) *There exists $a \in S_\sigma^{l_1, m_1}$ such that $a \sim \sum_{j=1}^{\infty} a_j$.*
- (ii) *If another a' satisfies $a' \sim \sum_{j=1}^{\infty} a_j$, then $a - a' \in S_\sigma^{-\infty}$.*

Proof. (ii) is obvious. Let us prove (i) for a sequence $(a_j)_{j \in \mathbb{N}^*}$ in $S_\sigma^{l_j, m_j}(\mathbb{R}^{2n}, L(E_z))$ and with $a \sim \sum_{j=1}^{\infty} a_j \in S_\sigma^{l_1, m_1}(\mathbb{R}^{2n}, L(E_z))$. The result will then follow for a sequence (b_j) in $S_\sigma^{l, m}$ by taking $b := T_{z,b,*}^{-1}(a)$ where $a_j := T_{z,b,*}(b_j)$. Define

$$a'_j(x, \vartheta) := \Delta_{p_j}(x, \vartheta) a_j(x, \vartheta)$$

where Δ_{p_j} is defined in the proof of Lemma 5.4.6 and (p_j) is a real sequence in $[1, +\infty[$. For any $j \in \mathbb{N}$, $a'_j - a_j \in S_{\sigma}^{-\infty}(\mathbb{R}^{2n}, L(E_z))$. Thus, the result will follow if we prove that for a specified sequence (p_j) and for any $N \geq 0$, there exists $k_0(N) \geq 2$ such that for any $k \geq k_0(N)$,

$$\sum_{j=k+1}^{\infty} q_{N,l_k,m_k}(a'_j) < \infty \quad (5.23)$$

where $q_{N,l_k,m_k} := \sup_{|\nu| \leq N} q_{\nu,l_k,m_k}$, and q_{ν,l_k,m_k} are the semi-norms of $S_{\sigma}^{l_k,m_k}(\mathbb{R}^{2n}, L(E_z))$. Indeed, with $\|\partial^{\nu} a'_j\|_{\infty} \leq q_{|\nu|,l_k,m_k}(a'_j)$ for $k \geq k_1(\nu)$, $a' := \sum_{j=1}^{\infty} a'_j$ is a well defined smooth function and we have then $a' - \sum_{j=1}^{k-1} a_j \in S_{\sigma}^{l_k,m_k}(\mathbb{R}^{2n}, L(E_z))$. Using Leibniz rule, we see that for any $2n$ -multi-index $\nu := (\alpha, \beta)$, and any $j \in \mathbb{N}^*$, there is $K_{\nu,j} > 0$ such that

$$\begin{aligned} \frac{1}{K_{\nu,j}} \|\partial^{\nu} a'_j(x, \vartheta)\|_{L(E_z)} &\leq \Delta_p(x, \vartheta) \langle x \rangle^{\sigma(l_j - |\alpha|)} \langle \vartheta \rangle^{m_j - |\beta|} \\ &\quad + \sum_{\nu' < \nu} |\partial^{\nu - \nu'} \Delta_p(x, \vartheta)| \langle x \rangle^{\sigma(l_j - |\alpha'|)} \langle \vartheta \rangle^{m_j - |\beta'|}. \end{aligned}$$

Let us suppose that $\sigma = 0$. The estimate (5.21) yields for any $N \geq 0, k \geq 2, j \geq k + 1$,

$$q_{N,l_k,m_k}(a'_j) \leq K_{N,j} \langle p_j \rangle^{m_j - m_{j-1}}$$

for a constant $K_{N,j} > 0$. If we now fix p_j as $p_j = (2^j \sup_{N \leq j} \{K_{N,j}, 1\})^{1/(m_{j-1} - m_j)}$, then we see that for any $N \geq 0, k \geq N + 2, j \geq k + 1$, we have $q_{N,l_k,m_k}(a'_j) \leq 2^{-j}$ and (5.23) is satisfied. Suppose now $\sigma \neq 0$. The estimate (5.22) yields for any $N \geq 0, k \geq 2, j \geq k + 1$,

$$q_{N,l_k,m_k}(a'_j) \leq K'_{N,j} \langle p_j \rangle^{r_j}$$

for a constant $K'_{N,j} > 0$ and with $r_j := \max\{m_j - m'_{j-1}, \sigma(l_j - l'_{j-1})\} < 0$. If we now fix p_j as $p_j = (2^j \sup_{N \leq j} \{K'_{N,j}, 1\})^{-r_j^{-1}}$, then we see that for any $N \geq 0, k \geq N + 2$, (5.23) is satisfied as for the case $\sigma = 0$. \square

5.4.2 Amplitudes and associated operators on $\mathcal{S}(\mathbb{R}^n, E_z)$

We shall see in this section amplitudes as generalizations of symbols of the type $S_{\sigma,z}^{l,m} := S_{\sigma}^{l,m}(\mathbb{R}^{2n}, L(E_z))$ where $z \in M$ is fixed. For each amplitude, a continuous operator from $\mathcal{S}(\mathbb{R}^n, E_z)$ into itself will be defined. Here the spaces $L(E_z)$ and E_z can simply be considered as $\mathcal{M}_n(\mathbb{C})$ and \mathbb{C}^n . The results in this section will be important for pseudodifferential operators on M in the next section.

Definition 5.4.10. An amplitude of order l, w, m and type $\sigma \in [0, 1], \kappa \geq 0$, is a smooth function $a \in C^{\infty}(\mathbb{R}^{3n}, L(E_z))$ such that for any $3n$ -multi-index $\nu = (\alpha, \beta, \gamma)$, there exists $C_{\nu} > 0$ such that

$$\|\partial^{(\alpha,\beta,\gamma)} a(x, \zeta, \vartheta)\|_{L(E_z)} \leq C_{\nu} \langle x \rangle^{\sigma(l - |\alpha + \beta|)} \langle \zeta \rangle^{w + \kappa|\alpha + \beta|} \langle \vartheta \rangle^{m - |\gamma|} \quad (5.24)$$

for any $(x, \zeta, \vartheta) \in \mathbb{R}^{3n}$. We note $\Pi_{\sigma,\kappa}^{l,w,m} := \Pi_{\sigma,\kappa}^{l,w,m}(\mathbb{R}^{3n}, L(E_z))$ the space of amplitudes of order l, w, m and type σ, κ .

Remark that $\Pi_{0,\kappa,z}^{l,w,m}$ is independent of l , we denote this space $\Pi_{0,\kappa,z}^{0,w,m}$. We note $\Pi_{\sigma,\kappa,z}^{-\infty,w} := \cap_{l,m} \Pi_{\sigma,\kappa,z}^{l,w,m}$. We set $\Pi_{\sigma,\kappa,z}^{\infty} := \cup_{l,w,m} \Pi_{\sigma,\kappa,z}^{l,w,m}$ and $\Pi_{\sigma,z}^{-\infty} := \cap_{l,m} \cup_{w,\kappa} \Pi_{\sigma,\kappa,z}^{l,w,m}$. We see that $\Pi_{\sigma,\kappa,z}^{l,w,m} \cdot \Pi_{\sigma,\kappa,z}^{l',w',m'} \subseteq \Pi_{\sigma,\kappa,z}^{l+l',w+w',m+m'}$ and $\Pi_{\sigma,\kappa,z}^{l,w,m} \subseteq \Pi_{\sigma,\kappa,z}^{l',w',m'}$ for $m \leq m'$, $w \leq w'$, and $l \leq l'$. Thus, $\Pi_{\sigma,\kappa,z}^{\infty}$ is a $*$ -algebra, which is trigraduated for $\sigma > 0$ and bigraduated for $\sigma = 0$. Note also that if $a \in \Pi_{\sigma,\kappa,z}^{l,w,m}$, then $\partial^{(\alpha,\beta,\gamma)} a \in \Pi_{\sigma,\kappa,z}^{l-|\alpha+\beta|,w+\kappa|\alpha+\beta|,m-|\gamma|}$.

Amplitudes and symbols in $S_{\sigma,z}^{l,m}$ are related by the following lemma:

Lemma 5.4.11. (i) For any $a \in \Pi_{\sigma,\kappa,z}^{l,w,m}$ we have $a_{\zeta=0} := (x, \vartheta) \mapsto a(x, 0, \vartheta)$ in $S_{\sigma,z}^{l,m}$.
(ii) For any $s \in S_{\sigma,z}^{l,m}$, the function $(x, \zeta, \vartheta) \mapsto s(x, \vartheta)$ is in $\Pi_{\sigma,0,z}^{l,0,m}$.
(iii) For any $f \in S_{\sigma}(\mathbb{R}^n)$, the function $(x, \zeta, \vartheta) \mapsto f(x) \text{Id}_{L(E_z)}$ is in $\Pi_{\sigma,0,z}^{0,0,0}$.

Proof. (i) follows from the fact that $\partial^\nu(a \circ P) = (\partial^{P(\nu)} a) \circ P$ where $P(x, \vartheta) := (x, 0, \vartheta)$.
(ii) Noting $Q(x, \zeta, \vartheta) := (x, \vartheta)$, the result follows from $\partial^{\alpha,\beta,\gamma}(s \circ Q) = \delta_{\beta,0}(\partial^{\alpha,\gamma}s) \circ Q$.
(iii) follows from (ii) and the fact that $(x, \vartheta) \mapsto f(x) \text{Id}_{L(E_z)} \in S_{\sigma,z}^{0,0,0}$. \square

As the spaces of symbols, the $\Pi_{\sigma,\kappa,z}^{l,w,m}$ are naturally Fréchet spaces:

Lemma 5.4.12. The following semi-norms on $\Pi_{\sigma,\kappa,z}^{l,w,m}$:

$$q_{(\alpha,\beta,\gamma)}^{l,w,m}(a) := \sup_{(x,\zeta,\vartheta) \in \mathbb{R}^{3n}} \langle x \rangle^{\sigma(|\alpha+\beta|-l)} \langle \zeta \rangle^{-w-\kappa|\alpha+\beta|} \langle \vartheta \rangle^{|\gamma|-m} \left\| \partial^{(\alpha,\beta,\gamma)} a(x, \zeta, \vartheta) \right\|_{L(E_z)}$$

determine a Fréchet topology on $\Pi_{\sigma,\kappa,z}^{l,w,m}$. The following inclusions are continuous for these topologies: $\Pi_{\sigma,\kappa,z}^{l,w,m} \cdot \Pi_{\sigma,\kappa,z}^{l',w',m'} \subseteq \Pi_{\sigma,\kappa,z}^{l+l',w+w',m+m'}$, $\Pi_{\sigma,\kappa,z}^{l,w,m} \subseteq \Pi_{\sigma,\kappa,z}^{l',w',m'}$ ($m \leq m'$, $w \leq w'$ and $l \leq l'$) and $\Pi_{\sigma,\kappa,z}^{-\infty,w} \subseteq \Pi_{\sigma,\kappa,z}^{l,w,m}$. Moreover, the last inclusion is dense when $\Pi_{\sigma,\kappa,z}^{l,w,m}$ has the topology of $\Pi_{\sigma,\kappa,z}^{l',w',m'}$ for $m < m'$ and $l < l'$.

Proof. The continuity results are straightforward. For the density result, we prove as in Lemma 5.4.6, that for any $a \in \Pi_{\sigma,\kappa,z}^{l,w,m}$ the sequence

$$a_p(x, \zeta, \vartheta) := (\rho(x/p))^{1-\delta_{\sigma,0}} \rho(\vartheta/p) a(x, \zeta, \vartheta) =: (1 - \Delta_p(x, \vartheta)) a(x, \zeta, \vartheta)$$

converges to a for the topology of $\Pi_{\sigma,\kappa}^{l',w',m'}(\mathbb{R}^{2n}, L(E_z))$ where $m' > m$ and $l' > l$. First note that the application $(x, \zeta, \vartheta) \mapsto (\rho(x/p))^{1-\delta_{\sigma,0}} \rho(\vartheta/p) \text{Id}_{L(E_z)}$ is an amplitude in $\Pi_{\sigma,0,z}^{-\infty,0}$. Thus, $(a_p)_{p \in \mathbb{N}^*}$ is a sequence in $\Pi_{\sigma,\kappa,z}^{-\infty,w}$. We define the function R_p such that $q_{(\alpha,\beta,\gamma)}^{l',w',m'}(a - a_p) = \sup_{(x,\zeta,\vartheta) \in \mathbb{R}^{3n}} R_p(x, \zeta, \vartheta)$, where $m' > m$ and $l' > l$. For a given $3n$ -multi-index $\nu := (\alpha, \beta, \gamma)$, we get with Leibniz rule, for a $K > 0$,

$$\begin{aligned} \frac{1}{K} R_p(x, \zeta, \vartheta) &\leq \Delta_p(x, \vartheta) \langle x \rangle^{\sigma(l-l')} \langle \vartheta \rangle^{m-m'} + \sum_{\nu' < \nu} |\partial^{\nu-\nu'} \Delta_p(x, \vartheta)| \\ &\quad \times \langle x \rangle^{\sigma(l-l'+|\alpha+\beta|-|\alpha'+\beta'|)} \langle \zeta \rangle^{\kappa(|\alpha'+\beta'|-|\alpha+\beta|)} \langle \vartheta \rangle^{m-m'+|\gamma|-|\gamma'|}. \end{aligned}$$

Suppose that $\sigma = 0$. In that case, $|\Delta_p(x, \vartheta)| \leq 1_{[p,+\infty]}(\vartheta)$ and if $\nu' < \nu$,

$$|\partial^{\nu-\nu'} \Delta_p(x, \vartheta)| \leq \delta_{\alpha,\alpha'} \delta_{\beta,\beta'} K_\gamma p^{-|\gamma|+|\gamma'|} 1_{[p,2p]}(\vartheta).$$

As a consequence we find $R_p(x, \zeta, \vartheta) = \mathcal{O}_{p \rightarrow \infty}(\langle p \rangle^{m-m'})$, as in Lemma 5.4.6. Suppose now $\sigma \neq 0$. In that case $|\Delta_p(x, \vartheta)| \leq 1_{F_p}(x, \vartheta)$ where $F_p := \mathbb{R}^{2n} - B_n(0, p) \times B_n(0, p)$ and if $\nu' < \nu$, for a constant $K_\nu > 0$

$$|\partial^{\nu-\nu'} \Delta_p(x, \vartheta)| \leq \delta_{\beta-\beta', 0} K_\nu 1_{[\text{sgn}(\alpha-\alpha')p, 2p]}(x) 1_{[\text{sgn}(\gamma-\gamma')p, 2p]}(\vartheta) p^{-|\nu|+|\nu'|}.$$

As a consequence, we find $R_p(x, \zeta, \vartheta) = \mathcal{O}_{p \rightarrow \infty}(\langle p \rangle^r)$ where $r := \max\{m - m', \sigma(l - l')\} < 0$ and the result follows. \square

We shall note Δ_ζ the differential operator $\sum_{i=1}^n \partial_{\zeta_i}^2$. The following formula is valid for any $\vartheta, \zeta \in \mathbb{R}^n$ and $p \in \mathbb{N}$,

$$\langle \vartheta \rangle^{2p} e^{2\pi i \langle \vartheta, \zeta \rangle} = (1 - (2\pi)^{-2} \Delta_\zeta)^p e^{2\pi i \langle \vartheta, \zeta \rangle} =: L_\zeta^p e^{2\pi i \langle \vartheta, \zeta \rangle}. \quad (5.25)$$

A computation shows that $(1 - (2\pi)^{-2} \Delta_\zeta)^p = \sum_{0 \leq |\beta| \leq p} c_{p, \beta} \partial_\zeta^{2\beta}$, where the summation is on n -multi-indices β and $c_{p, \beta} := \binom{p}{|\beta|} (-1)^{|\beta|} (2\pi)^{-2|\beta|} \beta!$. We shall also use the following useful formula valid for any $\vartheta \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^n \setminus \{0\}$ and $p \in \mathbb{N}$,

$$e^{2\pi i \langle \vartheta, \zeta \rangle} = \sum_{|\beta|=p} \lambda_\beta \frac{\zeta^\beta}{\|\zeta\|^{2p}} \partial_\vartheta^\beta e^{2\pi i \langle \vartheta, \zeta \rangle} =: M_\vartheta^{p, \zeta} e^{2\pi i \langle \vartheta, \zeta \rangle} \quad (5.26)$$

where $\lambda_\beta := \beta! (2\pi)^{-|\beta|} i^{|\beta|}$. We define ${}^t M_\vartheta^{p, \zeta} := \sum_{|\beta|=p} \lambda_\beta (-1)^p \frac{\zeta^\beta}{\|\zeta\|^{2p}} \partial_\vartheta^\beta$.

Definition 5.4.13. We note $\mathcal{O}_{f, z}$, where $f_1, f_2, f_3 : \mathbb{N}^{3n} \rightarrow \mathbb{R}$, and $f := (f_1, f_2, f_3)$, the space of smooth functions in $C^\infty(\mathbb{R}^{3n}, L(E_z))$ such that for any $3n$ -multi-index $\nu = (\alpha, \beta, \gamma)$, there is $C_\nu > 0$ such that

$$\|\partial^\nu a(x, \zeta, \vartheta)\|_{L(E_z)} \leq C_\nu \langle x \rangle^{f_1(\nu)} \langle \zeta \rangle^{f_2(\nu)} \langle \vartheta \rangle^{f_3(\nu)}$$

uniformly in $(x, \zeta, \vartheta) \in \mathbb{R}^{3n}$.

The vector space $\mathcal{O}_{f, z}$ has a natural family of seminorms q_ν^f given by the best constants C_ν in the previous estimate. With this family, $\mathcal{O}_{f, z}$ is a Fréchet space. Obviously, amplitudes in $\Pi_{\sigma, \kappa, z}^{l, w, m}$ form an $\mathcal{O}_{f, z}$ space where $f_1(\nu) := \sigma(l - |\alpha + \beta|)$, $f_2(\nu) := w + \kappa|\alpha + \beta|$ and $f_3(\nu) := m - |\gamma|$. For a given triple $f := (f_1, f_2, f_3)$ and $\rho \in \mathbb{R}$, we will note $f_{3, \rho, \alpha, \gamma} := \sup_\beta f_3(\alpha, \beta, \gamma) - \rho|\beta|$, $f_{2, \rho, \alpha, \beta} := \sup_\gamma f_2(\alpha, \beta, \gamma) - \rho|\gamma|$ and $f_{1, \rho, \alpha, \beta} := \sup_\gamma f_1(\alpha, \beta, \gamma) - \rho|\gamma|$.

Proposition 5.4.14. Let Γ a continuous linear operator on the space $\mathcal{S}(\mathbb{R}^{2n}, L(E_z))$, and $f := (f_1, f_2, f_3)$ a triple such that there exists $\rho < 1$ such that $f_{3, \rho, 0, 0} < \infty$.

(i) For any function $a \in \mathcal{O}_{f, z}$ the following antilinear form on $\mathcal{S}(\mathbb{R}^{2n}, L(E_z))$

$$\langle \mathfrak{Dp}_\Gamma(a), u \rangle := \int_{\mathbb{R}^{3n}} e^{2\pi i \langle \vartheta, \zeta \rangle} \text{Tr}(a(x, \zeta, \vartheta) \Gamma(u)^*(x, \zeta)) d\zeta d\vartheta dx$$

is in $\mathcal{S}'(\mathbb{R}^{2n}, L(E_z))$.

(ii) For any given $u \in \mathcal{S}(\mathbb{R}^{2n}, L(E_z))$, the linear form $L_{u, \Gamma} := a \mapsto \langle \mathfrak{Dp}_\Gamma(a), u \rangle$ is continuous on $\mathcal{O}_{f, z}$. In particular $L_{u, \Gamma}$ is continuous on any amplitude space $\Pi_{\sigma, \kappa, z}^{l, w, m}$.

Proof. (i) We have $\mathfrak{Op}_\Gamma(a) = I(a) \circ \Gamma$, where $I(a)$ is the antilinear form on $\mathcal{S}(\mathbb{R}^{2n}, L(E_z))$:

$$\langle I(a), u \rangle := \int_{\mathbb{R}^{3n}} e^{2\pi i \langle \vartheta, \zeta \rangle} \text{Tr}(a(x, \zeta, \vartheta) u^*(x, \zeta)) d\zeta d\vartheta dx.$$

We shall prove that $I(a) \in \mathcal{S}'(\mathbb{R}^{2n}, L(E_z))$, which will give the result. Let $u \in \mathcal{S}(\mathbb{R}^{2n}, L(E_z))$ and let us fix for now x and $\vartheta \in \mathbb{R}^n$. We can check that the map $\zeta \mapsto a(x, \zeta, \vartheta) u^*(x, \zeta)$ is in $\mathcal{S}(\mathbb{R}^n, L(E_z))$. As a consequence, with (5.25) and integration by parts, we get with $R(x, \vartheta) := \int_{\mathbb{R}^n} e^{2\pi i \langle \vartheta, \zeta \rangle} a(x, \zeta, \vartheta) u^*(x, \zeta) d\zeta$,

$$\begin{aligned} R(x, \vartheta) &= \int_{\mathbb{R}^n} e^{2\pi i \langle \vartheta, \zeta \rangle} \langle \vartheta \rangle^{-2p} (1 - (2\pi)^{-2} \Delta_\zeta)^p a(x, \zeta, \vartheta) u^*(x, \zeta) d\zeta \\ &= \sum_{0 \leq |\beta| \leq p} \sum_{\beta' \leq 2\beta} c_{p,\beta} \binom{2\beta}{\beta'} \langle \vartheta \rangle^{-2p} \int_{\mathbb{R}^n} e^{2\pi i \langle \vartheta, \zeta \rangle} (\partial^{(0,\beta',0)} a(x, \zeta, \vartheta)) (\partial^{(0,2\beta-\beta')} u^*(x, \zeta)) d\zeta. \end{aligned}$$

Thus, for any $x, \vartheta \in \mathbb{R}^n$, we get by fixing p such that $2(\rho - 1)p + f_{3,\rho,0,0} \leq -2n$ (this is possible since $\rho < 1$) that for any $N \in \mathbb{N}$,

$$\|R(x, \vartheta)\|_{L(E_z)} \leq C_p \langle \vartheta \rangle^{-2n} \int_{\mathbb{R}^n} \langle x, \zeta \rangle^{-N+r_p} d\zeta \sum_{0 \leq |\beta| \leq p} \sum_{\beta' \leq 2\beta} q_{0,\beta',0}^f(a) q_{N,(0,2\beta-\beta')}(u)$$

for a $C_p > 0$, where $r_p := \max_{|\beta'| \leq 2p} |f_1(0, \beta', 0)| + |f_2(0, \beta', 0)|$. If we now fix N such that $-N + r_p \leq -4n$, we see, using the inequality $\langle x, \zeta \rangle^{-2} \leq \langle x \rangle^{-1} \langle \zeta \rangle^{-1}$, that there is $C_{\rho,f} > 0$ such that

$$|\langle I(a), u \rangle| \leq C_{\rho,f} \sum_{0 \leq |\beta| \leq p} \sum_{\beta' \leq 2\beta} q_{0,\beta',0}^f(a) q_{N,(0,2\beta-\beta')}(u) \quad (5.27)$$

which yields the result.

(ii) The continuity of $L_{u,\Gamma}$ on $\mathcal{O}_{f,z}$ follows directly from (5.27) since $L_{u,\Gamma}(a) = \langle I(a), \Gamma(u) \rangle$. Since $\Pi_{\sigma,\kappa,z}^{l,w,m} = \mathcal{O}_{f,z}$ for a triple $f = (f_1, f_2, f_3)$ such that $f_{3,0,0,0} < \infty$, $L_{u,\Gamma}$ is continuous on any amplitude space. \square

For any amplitude a , we will also note $\mathfrak{Op}_\Gamma(a)$ the continuous linear map from $\mathcal{S}(\mathbb{R}^n, E_z)$ into $\mathcal{S}'(\mathbb{R}^n, E_z)$, associated to the tempered distribution $u \mapsto \langle \mathfrak{Op}_\Gamma(a), u \rangle$.

Remark 5.4.15. If $(M, \exp, E, d\mu, \psi)$ has a \mathcal{O}_M -bounded geometry, we saw that for any frame (z, \mathbf{b}) and $\lambda \in [0, 1]$, the $\Gamma_{\lambda,z,\mathbf{b}}$ maps are topological isomorphisms on $\mathcal{S}'(\mathbb{R}^{2n}, L(E_z))$. Thus, Lemma 5.4.14 implies that for a given $a \in \Pi_{\sigma,\kappa,z}^{l,w,m}$, we can define a family indexed by $\lambda \in [0, 1]$ of operators $\mathfrak{Op}_{\Gamma_{\lambda,z,\mathbf{b}}}(a)$ which are continuous from $\mathcal{S}(\mathbb{R}^n, E_z)$ into $\mathcal{S}'(\mathbb{R}^n, E_z)$.

Remark 5.4.16. Suppose that $(M, \exp, E, d\mu)$ has a \mathcal{S}_σ bounded geometry and that ψ is a \mathcal{O}_M -linearization. We deduce from (5.19) that if s is a symbol in $S_\sigma^{l,m}$ and $\lambda \in [0, 1]$, we have $(\mathfrak{Op}_\lambda(s))_{z,\mathbf{b}} = \mathfrak{Op}_{\Gamma_{\lambda,z,\mathbf{b}}}(\mu s_{z,\mathbf{b}})$ where (z, \mathbf{b}) is a frame, $s_{z,\mathbf{b}} := T_{z,\mathbf{b},*}(s)$ and $\mu s_{z,\mathbf{b}} := (x, \zeta, \vartheta) \mapsto \mu_{z,\mathbf{b}}(x) s_{z,\mathbf{b}}(x, \vartheta) \in \Pi_{\sigma,0,z}^{l,0,m}$. We will also note $\mu^{-1} s_{z,\mathbf{b}}(x, \zeta, \vartheta) := \mu_{z,\mathbf{b}}^{-1}(x) s_{z,\mathbf{b}}(x, \vartheta) \in \Pi_{\sigma,0,z}^{l,0,m}$.

We now establish a sufficient condition on Γ and a in order to have $\mathfrak{Op}_\Gamma(a)$ stable (and continuous) on $\mathcal{S}(\mathbb{R}^n, E_z)$. The result will be used to establish regularity of pseudodifferential operators.

Lemma 5.4.17. *Let Γ be a continuous linear operator on $\mathcal{S}(\mathbb{R}^{2n}, L(E_z))$ of the form $\Gamma = L_{\tau_1} \circ R_{\tau_2} \circ C_\Phi$, where $\tau_i \in \mathcal{O}_M(\mathbb{R}^{2n}, L(E_z))$ (for $1 \leq i \leq 2$), and $\Phi := (\pi_1, \psi) \in C^\infty(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ is such that $\psi \in \mathcal{O}_M(\mathbb{R}^{2n}, \mathbb{R}^n)$ and there exist $c, \varepsilon, r > 0$, such that for any $(x, \zeta) \in \mathbb{R}^{2n}$, $\langle \psi(x, \zeta) \rangle \geq c \langle x \rangle^\varepsilon \langle \zeta \rangle^{-r}$ and for any $x \in \mathbb{R}^n$, there is $c_x > 0$ such that $\langle \psi(x, \zeta) \rangle \geq c_x \langle \zeta \rangle^\varepsilon$ uniformly in $\zeta \in \mathbb{R}^n$.*

Suppose that $f = (f_1, f_2, f_3)$ is such that there exist $(\rho_1, \rho_2, \rho_3) \in \mathbb{R}^3$ such that $\rho_3 < 1$, $(r/\varepsilon)\rho_1 + \rho_2 < 1$ and for any $2n$ -multi-index μ , $f_{1,\rho_1,\mu} < \infty$, $f_{2,\rho_2,\mu} < \infty$, $f_{3,\rho_3,\mu} < \infty$ and for any n -multi-index α $f_{3,\rho_3,\alpha} := \sup_\gamma f_{3,\rho_3,\alpha,\gamma} < \infty$. Then for any function $a \in \mathcal{O}_{f,z}$, the operator $\mathfrak{Op}_\Gamma(a)$ is continuous from $\mathcal{S}(\mathbb{R}^n, E_z)$ into itself. In particular, this is the case for any amplitude $a \in \Pi_{\sigma,\kappa,z}^{l,w,m}$.

Proof. Let $u, v \in \mathcal{S}(\mathbb{R}^n, E_z)$. By definition, $\langle \mathfrak{Op}_\Gamma(a)(v), u \rangle = \mathfrak{Op}_\Gamma(a)(u \otimes \bar{v})$ and $\Gamma(K) = \tau_1(K \circ \Phi) \tau_2$. Noting $a'(x, \zeta, \vartheta) := \tau_1^*(x, \zeta) a(x, \zeta, \vartheta) \tau_2^*(x, \zeta)$, we obtain

$$\begin{aligned} \langle \mathfrak{Op}_\Gamma(a)(v), u \rangle &:= \int_{\mathbb{R}^{3n}} e^{2\pi i \langle \vartheta, \zeta \rangle} (a'(x, \zeta, \vartheta) v(\psi(x, \zeta)) | u(x)) d\zeta d\vartheta dx \\ &= \int_{\mathbb{R}^n} (g(x) | u(x)) dx \end{aligned}$$

where $g(x) := \int_{\mathbb{R}^{2n}} e^{2\pi i \langle \vartheta, \zeta \rangle} a'(x, \zeta, \vartheta) v \circ \psi(x, \zeta) d\zeta d\vartheta$.

A computation with the Faa di Bruno formula shows that for any $2n$ -multi-index ν , any $N \in \mathbb{N}$ and any $x \in \mathbb{R}^n$ there is $C_{x,N,\nu} > 0$ such that $\|\partial^\nu(v \circ \psi)(x, \zeta)\|_{E_z} \leq C_{x,N,\nu} \langle \zeta \rangle^{-N}$ uniformly in $\zeta \in \mathbb{R}^n$. As a consequence, the map $\zeta \mapsto \partial^{\alpha',0} a'(x, \zeta, \vartheta) \partial^{\alpha-\alpha'}(v \circ \psi)(x, \zeta)$ is in $\mathcal{S}(\mathbb{R}^n, E_z)$. We can thus successively integrate by parts in $g(x)$ so that for any $p \in \mathbb{N}^*$,

$$g(x) = \int_{\mathbb{R}^{2n}} e^{2\pi i \langle \vartheta, \zeta \rangle} \langle \vartheta \rangle^{-2p} L_\zeta^p(a'(v \circ \psi))(x, \zeta, \vartheta) d\zeta d\vartheta.$$

By taking p such that $(\rho_3 - 1)2p + c_0 \leq -2n$ where $c_\alpha := \sup_{\alpha' \leq \alpha} f_{3,\rho_3,\alpha'}$, we see that the previous integrand is absolutely integrable, and we can permute the order of integrations $d\zeta d\vartheta \rightarrow d\vartheta d\zeta$. Since all the successive ϑ -derivatives of $\langle \vartheta \rangle^{-2p} L_\zeta^p(a'(v \circ \psi))(x, \zeta, \vartheta)$ converge to 0 when $\langle \vartheta \rangle$ goes to infinity, we can then integrate by parts in ϑ so that for any $q \in \mathbb{N}$ and $p \geq p_0$

$$g(x) = \int_{\mathbb{R}^{2n}} e^{2\pi i \langle \vartheta, \zeta \rangle} \langle \zeta \rangle^{-2q} L_\vartheta^q(\langle \vartheta \rangle^{-2p} L_\zeta^p(a'(v \circ \psi)))(x, \zeta, \vartheta) d\zeta d\vartheta.$$

Noting $h_{p,q}$ the previous integrand, we see that for any n -multi-index α , $\partial^\alpha h_{p,q}$ is a linear combination of terms of the form

$$e^{2\pi i \langle \vartheta, \zeta \rangle} \langle \zeta \rangle^{-2q} \langle \vartheta \rangle^{-2p-|\gamma-\gamma'|} \partial^{\alpha',\beta',\gamma'} a' \partial^{\alpha-\alpha',\beta-\beta'} v \circ \psi$$

where $|\gamma| \leq 2p$, $\gamma' \leq \gamma$, $|\beta| \leq 2q$, $\beta' \leq \beta$ and $\alpha' \leq \alpha$. A computation with the Faa di Bruno formula shows that for any $2n$ -multi-index ν there is $r_\nu \in \mathbb{N}^*$ such that for any $N > 0$, there is $C_{\nu,N} > 0$ such that for any $w \in \mathcal{S}(\mathbb{R}^n, E_z)$ and any $(x, \zeta) \in \mathbb{R}^{2n}$, $\|\partial^\nu(w \circ \psi)(x, \zeta)\|_{E_z} \leq C_{\nu,N} \langle x, \zeta \rangle^{r_\nu-N} \langle \zeta \rangle^{r_\nu+(r/\varepsilon)N} \sum_{|\nu'| \leq |\nu|} q_{[N/\varepsilon]+1,\nu'}(w)$. Moreover, we check that there is $K_{\alpha,p} > 0$ such that

$$\left\| \partial^{(\alpha',\beta',\gamma')} a'(x, \zeta, \vartheta) \right\|_{L(E_z)} \leq C_{\alpha,p,q} \langle x \rangle^{K_{\alpha,p}+\rho_1 2q} \langle \zeta \rangle^{K_{\alpha,p}+\rho_2 2q} \langle \vartheta \rangle^{c_\alpha+\rho_3 2p}.$$

As a consequence, we get the estimate

$$\|\partial^\alpha h_{p,q}\| \leq C_{\alpha,p,q,N} \langle x \rangle^{K'_{\alpha,p} + \rho_1 2q - N} \langle \zeta \rangle^{K'_{\alpha,p} + (\rho_2 - 1)2q + (r/\varepsilon)N} \langle \vartheta \rangle^{c_\alpha + (\rho_3 - 1)2p} \sum_{|\nu'| \leq |\nu|} q_{[N/\varepsilon] + 1, \nu'}(v).$$

or equivalently, replacing $K'_{\alpha,p} + \rho_1 2q - N$ by $-N$,

$$\|\partial^\alpha h_{p,q}\| \leq C_{\alpha,p,q,N} \langle x \rangle^{-N} \langle \zeta \rangle^{K''_{\alpha,p} + (\rho_2 - 1 + (r/\varepsilon)\rho_1)2q + (r/\varepsilon)N} \langle \vartheta \rangle^{c_\alpha + (\rho_3 - 1)2p} \sum_{|\nu'| \leq |\nu|} q_{[N + K'_{\alpha,p} + \rho_1 2q/\varepsilon] + 1, \nu'}(v).$$

Fixing now, for a given N, p such that $(\rho_3 - 1)2p + c_\alpha \leq -2n$ and q such that $K''_{\alpha,p} + (\rho_2 - 1 + (r/\varepsilon)\rho_1)2q + (r/\varepsilon)N \leq -2n$, we obtain the result. \square

The following lemma gives a characterization of smoothing kernels in the cases $\sigma = 0$ and $\sigma \neq 0$. If s is in a space of symbols and Γ is a continuous linear map on $\mathcal{S}(\mathbb{R}^{2n}, L(E_z))$, we will note $\mathfrak{Op}_\Gamma(s) := \mathfrak{Op}_\Gamma((x, \zeta, \vartheta) \mapsto s(x, \vartheta))$. We shall use the Fréchet space $\mathcal{O}_{\sigma, f, z}^{l, m}$ of smooth functions a in $C^\infty(\mathbb{R}^{3n}, L(E_z))$ such that for any $\nu := (\mu, \gamma) \in \mathbb{N}^{2n} \times \mathbb{N}^n$

$$\|\partial^\nu a(x, \zeta, \vartheta)\|_{L(E_z)} \leq C_\nu \langle x \rangle^{\sigma(l + f_1(\mu))} \langle \zeta \rangle^{f_2(\nu)} \langle \vartheta \rangle^{m + f_3(\mu)}.$$

We will note $\mathcal{O}_{0, f, z}^{l, m} =: \mathcal{O}_{f_2, f_3, z}^m$. Clearly, $\mathfrak{Op}_\Gamma(a)$ (see Lemma 5.4.14) is defined as an antilinear form on $\mathcal{S}(\mathbb{R}^{2n}, L(E_z))$ whenever $a \in \mathcal{O}_{f, z}^{l, m}$ with $m + f_3(0) < -n$. We note F the set of functions $f_2 : \mathbb{N}^{2n} \rightarrow \mathbb{R}$ such that there is $\rho < 1$ such that for any $(\alpha, \beta) \in \mathbb{N}^{2n}$ $f_{2, \rho, \alpha, \beta} := \sup_\gamma f_2(\alpha, \beta, \gamma) - \rho|\gamma| < \infty$.

Lemma 5.4.18. *Let $K \in \mathcal{S}'(\mathbb{R}^{2n}, L(E_z))$, and Γ a topological isomorphism on $\mathcal{S}(\mathbb{R}^{2n}, L(E_z))$ of the form $\Gamma = L_{\tau_1} \circ R_{\tau_2} \circ C_\Phi$ with $\tau_1, \tau_2 \in \mathcal{O}_M^\times(\mathbb{R}^{2n}, GL(E_z))$, $\Phi \in \mathcal{O}_M^\times(\mathbb{R}^{2n}, \mathbb{R}^{2n})$. Then*

(i) *Case $\sigma = 0$. The following are equivalent:*

(i-1) *There is $f_3 : \mathbb{N}^{2n} \rightarrow \mathbb{R}$ such that for any $m \leq -f_3(0) - 2n$, there exist $f_{2,m} \in F$, $a_m \in \mathcal{O}_{f_{2,m}, f_3, z}^m$ such that $K = \mathfrak{Op}_\Gamma(a_m)$.*

(i-2) *$K \in C^\infty(\mathbb{R}^{2n}, L(E_z))$ and for any $2n$ -multi-index ν , $N \in \mathbb{N}$, there is $C_{\nu, N} > 0$ such that for any $(x, \zeta) \in \mathbb{R}^{2n}$, $\|\partial^\nu K_\Gamma(x, \zeta)\|_{L(E_z)} \leq C_{\nu, N} \langle \zeta \rangle^{-N}$, where $K_\Gamma := K \circ \Gamma = \tilde{\tau}_1 K \circ \Phi \tilde{\tau}_2 |J(\Phi)|$.*

(i-3) *There is $s \in S_{0, z}^{-\infty}$ such that $K = \mathfrak{Op}_\Gamma(s)$.*

(ii) *Case $\sigma > 0$. The following are equivalent:*

(ii-1) *There is $f_1, f_3 : \mathbb{N}^{2n} \rightarrow \mathbb{R}$ such that for any $m \leq -f_3(0) - 2n$, there exist $f_{2,m} \in F$ and $a_m \in \mathcal{O}_{\sigma, f_1, f_{2,m}, f_3, z}^{m, m}$ such that $K = \mathfrak{Op}_\Gamma(a_m)$.*

(ii-2) *$K \in \mathcal{S}(\mathbb{R}^{2n}, L(E_z))$.*

(ii-3) *There is $s \in S_z^{-\infty}$ such that $K = \mathfrak{Op}_\Gamma(s)$.*

Proof. (i) The implication (i-3) \Rightarrow (i-1) is trivial. We will prove (i-1) \Rightarrow (i-2) \Rightarrow (i-3). Suppose (i-1). Thus, for any $m \leq -2n - f_3(0)$, there is $f_{2,m} \in F$, $a_m \in \mathcal{O}_{f_{2,m}, f_3, z}^m$ such that for any $u \in \mathcal{S}(\mathbb{R}^{2n}, L(E_z))$,

$$\langle K \circ \Gamma^{-1}, u \rangle = \int_{\mathbb{R}^{3n}} e^{2\pi i \langle \vartheta, \zeta \rangle} \text{Tr}(a_m(x, \zeta, \vartheta) u^*(x, \zeta)) d\zeta d\vartheta dx.$$

Since $m \leq -2n - f_3(0)$, the preceding integral is absolutely convergent and we can permute the order of integration. As a consequence, we get $\langle K \circ \Gamma^{-1}, u \rangle = \int_{\mathbb{R}^{2n}} \text{Tr} (U_m(x, \zeta) u^*(x, \zeta)) d\zeta dx$ where $U_m(x, \zeta) := \int_{\mathbb{R}^n} e^{2\pi i \langle \vartheta, \zeta \rangle} a_m(x, \zeta, \vartheta) d\vartheta$, we check easily that U_m is a continuous function on \mathbb{R}^{2n} , so we deduce that $U_m =: U$ is independent of m and $K \circ \Gamma^{-1}$ is a distribution which is continuous function equal to U . Noting $b_m := e^{2\pi i \langle \vartheta, \zeta \rangle} a_m(x, \zeta, \vartheta)$ we see that for any $2n$ -multi-index $\mu := (\alpha, \beta)$, $\partial_{x, \zeta}^\mu b_m = e^{2\pi i \langle \vartheta, \zeta \rangle} \sum_{\beta' \leq \beta} \binom{\beta}{\beta'} (2\pi i \vartheta)^{\beta - \beta'} \partial^{\alpha, \beta', 0} a_m$ and we have then the estimates

$$\|\partial^\mu b_m\| \leq C_{\mu, m} \langle \zeta \rangle^{\sup_{\beta' \leq \beta} f_{2, m}(\alpha, \beta', 0)} \langle \vartheta \rangle^{m + c_\mu}$$

where $c_\mu = \sup_{\beta' \leq \beta} f_3(\alpha, \beta') + |\beta|$. Defining $m_\mu := -2n - \sup_{|\mu'| \leq |\mu|} c_{\mu'}$, we see that U is smooth and

$$\partial^\mu U = \int_{\mathbb{R}^{2n}} \partial^\mu b_{m_\mu} d\vartheta = \sum_{\beta' \leq \beta} \binom{\beta}{\beta'} (2\pi i)^{|\beta - \beta'|} \int_{\mathbb{R}^n} e^{2\pi i \langle \vartheta, \zeta \rangle} \vartheta^{\beta - \beta'} \partial^{\alpha, \beta', 0} a_{m_\mu}(x, \zeta, \vartheta) d\vartheta.$$

All the ϑ -derivatives of $\vartheta \mapsto \vartheta^{\beta - \beta'} \partial^{\alpha, \beta', 0} a_{m_\mu}(x, \zeta, \vartheta)$ converge to zero when $\|\vartheta\| \rightarrow \infty$ so we can integrate by parts in ϑ so that for any $p \in \mathbb{N}$:

$$\partial^\mu U = \sum_{\beta' \leq \beta} \binom{\beta}{\beta'} (2\pi i)^{|\beta - \beta'|} \int_{\mathbb{R}^n} e^{2\pi i \langle \vartheta, \zeta \rangle} \langle \zeta \rangle^{-2p} L_\vartheta^p (\vartheta^{\beta - \beta'} \partial^{\alpha, \beta', 0} a_{m_\mu})(x, \zeta, \vartheta) d\vartheta.$$

Since $a_{m_\mu} \in \mathcal{O}_{f_{2, m_\mu}, f_{3, z}}^{m_\mu}$ and $f_{2, m_\mu, \rho_\mu, \lambda} < \infty$ for a $\rho_\mu < 1$, we see that the integrand h_p of the previous integral satisfies the estimate

$$\|h_p(x, \zeta, \vartheta)\| \leq C_{p, \mu} \langle \zeta \rangle^{-2p + \sup_{\beta' \leq \beta} f_{2, m_\mu, \rho_\mu, \alpha, \beta'} + 2p\rho_\mu} \langle \vartheta \rangle^{-2n}.$$

Given $N > 0$ and fixing p such that $(\rho_\mu - 1)2p + \sup_{\beta' \leq \beta} f_{2, m_\mu, \rho_\mu, \alpha, \beta'} \leq -N$, we finally obtain that $K \circ \Gamma^{-1} = U$ is smooth and satisfies for any $\mu \in \mathbb{N}^{2n}$ and $N > 0$, $\|\partial^\mu K \circ \Gamma^{-1}(x, \zeta)\|_{L(E_z)} \leq C_{\mu, N} \langle \zeta \rangle^{-N}$. We also have for any $u \in \mathcal{S}(\mathbb{R}^{2n}, L(E_z))$, $\langle K, u \rangle = \langle U, \Gamma(u) \rangle = \int_{\mathbb{R}^{2n}} \text{Tr}(U'(x, \zeta) u^* \circ \Phi(x, \zeta)) dx d\zeta$ where $U'(x, \zeta) := \tau_1^*(x, \zeta) U(x, \zeta) \tau_2^*(x, \zeta)$. Using the change of variables provided by the diffeomorphism Φ , we get $\langle K, u \rangle = \int_{\mathbb{R}^{2n}} \text{Tr}(K(x, y) u^*(x, y)) dx dy$ where $K(x, y) := (|J(\Phi^{-1})|(x, y)) U' \circ \Phi^{-1}(x, y)$. The result follows.

Suppose now (i-2). It is not difficult to see that \mathcal{F}_P sends $S_{0, z}^{-\infty}$ (seen as a subspace of $\mathcal{S}'(\mathbb{R}^{2n}, L(E_z))$) into $S_{0, z}^{-\infty}$. In particular, we have $s := \mathcal{F}_P(K_\Gamma) \in S_{0, z}^{-\infty}$. A computation shows that $\langle K, u \rangle = \langle \mathfrak{Op}_\Gamma(s), u \rangle$ for any $u \in \mathcal{S}(\mathbb{R}^{2n}, L(E_z))$.

(ii) Suppose (i-1). Following the proof of (i), we see that it is sufficient to prove that U is in $\mathcal{S}(\mathbb{R}^{2n}, L(E_z))$, where $U(x, \zeta) := \int_{\mathbb{R}^n} e^{2\pi i \langle \vartheta, \zeta \rangle} a_m(x, \zeta, \vartheta) d\vartheta$ (independent of m). Let us fix $N > 0$. For any $2n$ -multi-index $\mu := (\alpha, \beta)$, $\partial_{x, \zeta}^\mu b_m = e^{2\pi i \langle \vartheta, \zeta \rangle} \sum_{\beta' \leq \beta} \binom{\beta}{\beta'} (2\pi i \vartheta)^{\beta - \beta'} \partial^{\alpha, \beta', 0} a_m$ and we have the estimates

$$\|\partial^\mu b_m\| \leq C_{\mu, m} \langle x \rangle^{\sigma m + \sigma d_\mu} \langle \zeta \rangle^{\sup_{\beta' \leq \beta} f_{2, m}(\alpha, \beta', 0)} \langle \vartheta \rangle^{m + c_\mu}$$

where $c_\mu = \sup_{\beta' \leq \beta} f_3(\alpha, \beta') + |\beta|$ and $d_\mu := \sup_{\beta' \leq \beta} f_1(\alpha, \beta')$. Defining

$$m_{\mu, N} := \min \left\{ -2n - \sup_{|\mu'| \leq |\mu|} c_{\mu'}, -N/\sigma - \sup_{|\mu'| \leq |\mu|} d_{\mu'} \right\}$$

we see that U is smooth and

$$\partial^\mu U = \int_{\mathbb{R}^{2n}} \partial^\mu b_{m_\mu, N} d\vartheta = \sum_{\beta' \leq \beta} \binom{\beta}{\beta'} (2\pi i)^{|\beta - \beta'|} \int_{\mathbb{R}^n} e^{2\pi i \langle \vartheta, \zeta \rangle} \vartheta^{\beta - \beta'} \partial^{\alpha, \beta', 0} a_{m_\mu, N}(\mathbf{x}, \zeta, \vartheta) d\vartheta.$$

All the ϑ -derivatives of $\vartheta \mapsto \vartheta^{\beta - \beta'} \partial^{\alpha, \beta', 0} a_{m_\mu, N}(\mathbf{x}, \zeta, \vartheta)$ converge to zero when $\|\vartheta\| \rightarrow \infty$ so we can integrate by parts in ϑ so that for any $p \in \mathbb{N}$:

$$\partial^\mu U = \sum_{\beta' \leq \beta} \binom{\beta}{\beta'} (2\pi i)^{|\beta - \beta'|} \int_{\mathbb{R}^n} e^{2\pi i \langle \vartheta, \zeta \rangle} \langle \zeta \rangle^{-2p} L_\vartheta^p (\vartheta^{\beta - \beta'} \partial^{\alpha, \beta', 0} a_{m_\mu, N})(\mathbf{x}, \zeta, \vartheta) d\vartheta.$$

Since $a_{m_\mu, N} \in \mathcal{O}_{\sigma, f_1, f_2, m_\mu, N, f_3, z}^{m_\mu, N, m_\mu, N}$ and $f_{2, m_\mu, N, \rho_\mu, N, \lambda} < \infty$ for a $\rho_\mu, N < 1$, we see that the integrand h_p of the previous integral satisfies the estimate

$$\|h_p(\mathbf{x}, \zeta, \vartheta)\| \leq C_{p, \mu, N} \langle \mathbf{x} \rangle^{-N} \langle \zeta \rangle^{-2p + \sup_{\beta' \leq \beta} f_{2, m_\mu, N, \rho_\mu, N, \alpha, \beta'} + 2p\rho_\mu, N} \langle \vartheta \rangle^{-2n}.$$

Fixing p such that $(\rho_\mu, N - 1)2p + \sup_{\beta' \leq \beta} f_{2, m_\mu, N, \rho_\mu, N, \alpha, \beta'} \leq -N$, we finally obtain the following estimate $\|\partial^\mu U\|_{L(E_z)} \leq C_{\mu, N} \langle \mathbf{x} \rangle^{-N} \langle \zeta \rangle^{-N}$, which yields (i-2). The other implications are straightforward. \square

Corollary 5.4.19. *Same hypothesis. We have (for $\sigma = 0$ or $\sigma > 0$), $\mathfrak{Op}_\Gamma(S_{\sigma, z}^{-\infty}) = \cap_{l, m} \cup_{w, \kappa} \mathfrak{Op}_\Gamma(\Pi_{\sigma, \kappa, z}^{l, w, m}) = \mathfrak{Op}_\Gamma(\Pi_{\sigma, z}^{-\infty})$.*

Lemma 5.4.20. *Let $u \in S(\mathbb{R}^{2n}, L(E_z))$ and β a n -multi-index.*

(i) *For any triple $f := (f_1, f_2, f_3)$ such that there exists $\rho < 1$ such that for any $2n$ -multi-index (α, γ) , $f_{3, \rho, \alpha, \gamma} < \infty$, the following linear forms are continuous on $\mathcal{O}_{f, z}$*

$$R_{\beta, u} : a \mapsto \int_{\mathbb{R}^{3n}} \zeta^\beta e^{2\pi i \langle \vartheta, \zeta \rangle} \text{Tr}(a(\mathbf{x}, \zeta, \vartheta) u(\mathbf{x}, \zeta)) d\zeta d\vartheta d\mathbf{x},$$

$$S_{\beta, u} : a \mapsto (i/2\pi)^{|\beta|} \int_{\mathbb{R}^{3n}} e^{2\pi i \langle \vartheta, \zeta \rangle} \text{Tr}(\partial_\vartheta^\beta a(\mathbf{x}, \zeta, \vartheta) u(\mathbf{x}, \zeta)) d\zeta d\vartheta d\mathbf{x}.$$

(ii) $R_{\beta, u} = S_{\beta, u}$ on any $\Pi_{\sigma, \kappa, z}^{l, w, m}$ space.

Proof. (i) The continuity of $R_{\beta, u}$ is a direct consequence of Proposition 5.4.14 since $R_{\beta, u} = L_{u_\beta, \text{Id}}$ where $u_\beta(\mathbf{x}, \zeta) := \zeta^\beta u(\mathbf{x}, \zeta)$. Suppose that ν_0 is a $3n$ -multi-index, we denote $f^{\nu_0} := \nu \mapsto f(\nu + \nu_0)$. A computation shows for any ρ , and n -multi-indices α, γ , $f_{3, \rho, \alpha, \gamma}^{\nu_0} \leq f_{3, \rho, \alpha + \alpha_0, \gamma + \gamma_0} + \rho|\beta_0|$. Thus if there is $\rho < 1$ such that for any $2n$ -multi-index (α, γ) , $f_{3, \rho, \alpha, \gamma} < \infty$, then for any $2n$ -multi-index (α, γ) , $f_{3, \rho, \alpha, \gamma}^{\nu_0} < \infty$. If $a \in \mathcal{O}_{f, z}$ then $\partial^{\nu_0} a \in \mathcal{O}_{f^{\nu_0}, z}$ and the linear map $a \mapsto \partial^{\nu_0} a$ is continuous. As a consequence, since $S_{\beta, u} = L_{u, \text{Id}} \circ D_\beta$, where $D_\beta := (i/2\pi)^\beta \partial_\vartheta^\beta$, the continuity of $S_{\beta, u}$ on $\mathcal{O}_{f, z}$ follows from Proposition 5.4.14.

(ii) The equality is easily obtained on $\Pi_{\sigma, \kappa, z}^{-\infty, w}$ by an integration by parts in ϑ and permutations of the order of integration $d\zeta d\vartheta \rightarrow d\vartheta d\zeta$ in $R_{\beta, u}(a)$ (authorized for $a \in \Pi_{\sigma, \kappa, z}^{-\infty, w}$). The result now follows from (i) and the density result of Lemma 5.4.12. \square

If $N \geq 1$ and β, γ , n -multi-indices, we denote for any amplitude $a \in \Pi_{\sigma, \kappa, z}^{l, w, m}$, the smooth function $a_{\beta, \gamma, N}$ as $a_{\beta, \gamma, N}(\mathbf{x}, \zeta, \vartheta) := \int_0^1 (1 - t)^N (\partial^{(0, \beta, \gamma)} a)(\mathbf{x}, t\zeta, \vartheta) dt$. It is straightforward to check that the linear map $a \mapsto a_{\beta, \gamma, N}$ is continuous from $\Pi_{\sigma, \kappa, z}^{l, w, m}$ into $\Pi_{\sigma, \kappa, z}^{l - |\beta|, |w| + \kappa|\beta|, m - |\gamma|}$.

The following lemma shows that λ -quantization of amplitudes and symbols yields the same operators. This result of “reduction” of amplitudes to symbols will be important for Theorem 5.4.30 and thus, for a λ -invariant definition of pseudodifferential operators.

Lemma 5.4.21. (i) For any $a \in \Pi_{\sigma,\kappa,z}^{l,w,m}$, $(\partial^{0,\beta,\beta}a)_{\zeta=0} \in S_{\sigma,z}^{l-|\beta|,m-|\beta|}$ for any n -multi-index β .
(ii) Let Γ be as in Lemma 5.4.18 and let $a \in \Pi_{\sigma,\kappa,z}^{l,w,m}$. Then for any symbol $s \in S_{\sigma,z}^{l,m}$ such that $s \sim \sum_{\beta} \frac{(i/2\pi)^{|\beta|}}{\beta!} (\partial^{0,\beta,\beta}a)_{\zeta=0}$, there is $r \in S_{\sigma,z}^{-\infty}$ such that $\mathfrak{Op}_{\Gamma}(a) = \mathfrak{Op}_{\Gamma}(s+r)$. In particular there exists a unique symbol $s(a) \in S_{\sigma,z}^{l,m}$ such that $\mathfrak{Op}_{\Gamma}(a) = \mathfrak{Op}_{\Gamma}(s(a))$. Moreover, we have $s(a) \sim \sum_{\beta} \frac{(i/2\pi)^{|\beta|}}{\beta!} (\partial^{0,\beta,\beta}a)_{\zeta=0}$.
(iii) Suppose that $(M, \exp, E, d\mu)$ has a S_{σ} -bounded geometry and ψ is a \mathcal{O}_M -linearization. Let $a \in \Pi_{\sigma,\kappa,z}^{l,w,m}$, $\lambda \in [0, 1]$ and (z, \mathbf{b}) be given a frame. Then there exists a unique symbol $s_{\lambda}(a) \in S_{\sigma}^{l,m}$ such that $\mathfrak{Op}_{\Gamma_{\lambda,z,\mathbf{b}}}(a) = (\mathfrak{Op}_{\lambda}(s_{\lambda}(a)))_{z,\mathbf{b}}$. Moreover, we have $T_{z,\mathbf{b},*}(s_{\lambda}(a)) \sim \sum_{\beta} \frac{(i/2\pi)^{|\beta|}}{\beta!} \mu^{-1} (\partial^{0,\beta,\beta}a)_{\zeta=0}$.

Proof. (i) is a direct consequence of Lemma 5.4.11 (i).

(ii) Using a Taylor expansion of a at $\zeta = 0$, we find that for any $u \in \mathcal{S}(\mathbb{R}^{2n}, L(E_z))$, $N \in \mathbb{N}^*$, $\langle \mathfrak{Op}_{\Gamma}(a), u \rangle = \sum_{0 \leq |\beta| \leq N} I_{\beta} + \sum_{|\beta|=N+1} \frac{N+1}{\beta!} R_{\beta,N}$ where

$$I_{\beta} := \int_{\mathbb{R}^{3n}} \zeta^{\beta} e^{2\pi i \langle \vartheta, \zeta \rangle} \text{Tr} \left(\frac{1}{\beta!} (\partial^{(0,\beta,0)} a)_{\zeta=0}(\mathbf{x}, \vartheta) \Gamma(u)^*(\mathbf{x}, \zeta) \right) d\zeta d\vartheta d\mathbf{x},$$

$$R_{\beta,N} := \int_{\mathbb{R}^{3n}} \zeta^{\beta} e^{2\pi i \langle \vartheta, \zeta \rangle} \text{Tr} (a_{\beta,0,N}(\mathbf{x}, \zeta, \vartheta) \Gamma(u)^*(\mathbf{x}, \zeta)) d\zeta d\vartheta d\mathbf{x}.$$

We get from Lemma 5.4.20 (ii),

$$I_{\beta} = \int_{\mathbb{R}^{3n}} e^{2\pi i \langle \vartheta, \zeta \rangle} \text{Tr} \left(\frac{(i/2\pi)^{|\beta|}}{\beta!} (\partial^{(0,\beta,\beta)} a)_{\zeta=0}(\mathbf{x}, \vartheta) \Gamma(u)^*(\mathbf{x}, \zeta) \right) d\zeta d\vartheta d\mathbf{x}.$$

Let $s \in S_{\sigma,z}^{l,m}$ be a symbol such that $s \sim \sum_{\beta} \frac{(i/2\pi)^{|\beta|}}{\beta!} (\partial^{0,\beta,\beta}a)_{\zeta=0}$. Then noting $s_N := s - \sum_{|\beta| \leq N} \frac{(i/2\pi)^{|\beta|}}{\beta!} (\partial^{0,\beta,\beta}a)_{\zeta=0} \in S_{\sigma,z}^{l-(N+1),m-(N+1)}$, we find with Lemma 5.4.20 (ii) that $\mathfrak{Op}_{\Gamma}(a-s) = \mathfrak{Op}_{\Gamma}(r_N)$ where

$$r_N := \sum_{|\beta|=N+1} \frac{(N+1)(i/2\pi)^{N+1}}{\beta!} a_{\beta,\beta,N} - s_N.$$

We check that $r_N \in \Pi_{\sigma,\kappa,z}^{l-(N+1),w_N,m-(N+1)}$ where $w_N = |w| + \kappa(N+1)$. Corollary 5.4.19 applied to $\mathfrak{Op}_{\Gamma}(a-s)$ now implies that there is $r \in S_{\sigma,z}^{-\infty}$ such that $\mathfrak{Op}_{\Gamma}(a) = \mathfrak{Op}_{\Gamma}(s+r)$. As a consequence, there exists $s(a) \in S_{\sigma,z}^{l,m}$ such that $\mathfrak{Op}_{\Gamma}(a) = (\mathfrak{Op}_{\Gamma}(s(a)))$. The unicity is a direct consequence of the fact that $\mathfrak{Op}_{\Gamma} = \Gamma^* \circ \mathcal{F}_P^*$ on $\mathcal{S}'(\mathbb{R}^{2n}, L(E_z))$.

(iii) Direct consequence of (ii) and that fact that $(\mathfrak{Op}_{\lambda}(s))_{z,\mathbf{b}} = \mathfrak{Op}_{\Gamma_{\lambda,z,\mathbf{b}}}(\mu_{z,\mathbf{b}} s_{z,\mathbf{b}})$. \square

5.4.3 S_{σ} -linearizations

In order to have a full symbol-operator isomorphism, a polynomial control at infinity on the linearization is not enough. As we shall see, a stronger, “amplitude-like” control on the $\psi_z^{\mathbf{b}}$ maps and a local equivalent of the $P_{x,\xi}$ parallel transport linear isomorphisms (see Remark 5.3.3)

appears to be crucial for pseudodifferential calculus on (M, \exp, E) and the λ -invariance (see Theorem 5.4.30).

We define $H_{\sigma, \kappa}^w(\mathfrak{E})$ (resp. $E_{\sigma, \kappa}^{w'}(\mathfrak{E})$), where $w \in \mathbb{R}$, $\sigma \in [0, 1]$ and $\kappa \geq 0$, as the space of smooth functions g from \mathbb{R}^{2n} into \mathfrak{E} such that for any $2n$ -multi-index ν , there exists $C_\nu > 0$ such that for any $(x, \zeta) \in \mathbb{R}^{2n}$, $\|\partial^\nu g(x, \zeta)\| \leq C_\nu \langle x \rangle^{-\sigma(|\nu|-1)} \langle \zeta \rangle^{w+\kappa(|\nu|-1)}$ (if $\nu \neq 0$) (resp. $\|\partial^\nu g(x, \zeta)\| \leq C_\nu \langle x \rangle^{-\sigma|\nu|} \langle \zeta \rangle^{w+\kappa|\nu|}$). We note $H_{\sigma, \kappa}(\mathfrak{E}) := \cup_{w \in \mathbb{R}} H_{\sigma, \kappa}^w(\mathfrak{E})$, $H_\sigma(\mathfrak{E}) := \cup_{\kappa \geq 0} H_{\sigma, \kappa}(\mathfrak{E})$, $E_{\sigma, \kappa}(\mathfrak{E}) = \cup_{w \in \mathbb{R}} E_{\sigma, \kappa}^w(\mathfrak{E})$ and $E_\sigma(\mathfrak{E}) = \cup_{\kappa \geq 0} E_{\sigma, \kappa}(\mathfrak{E})$. Remark that by Leibniz rule, $E_{\sigma, \kappa}(\mathbb{R})$ and $E_{\sigma, \kappa}(\mathcal{M}_p(\mathbb{R}))$ are \mathbb{R} -algebras (graduated by the parameter w) while $E_{\sigma, \kappa, z} := E_{\sigma, \kappa}(L(E_z))$ is a \mathbb{C} -algebra (under pointwise matricial product). Thus, if $P \in E_{\sigma, \kappa}(\mathcal{M}_p(\mathbb{R}))$, then $\det P \in E_{\sigma, \kappa}(\mathbb{R})$. Note also that $f \in H_{\sigma, \kappa}(\mathfrak{E})$ if and only if for any $i \in \{1, \dots, 2n\}$, $\partial_i f \in E_{\sigma, \kappa}(\mathfrak{E})$. In particular, $f \in H_{\sigma, \kappa}(\mathbb{R}^p)$ if and only if $df := (x, \zeta) \mapsto (df)_{x, \zeta}$ is in $E_{\sigma, \kappa}(\mathcal{M}_{p, 2n}(\mathbb{R}))$. As a consequence, if $f \in H_{\sigma, \kappa}(\mathbb{R}^{2n})$, its Jacobian determinant $J(g)$ is in $E_{\sigma, \kappa}(\mathbb{R})$. Note that any function in $E_{\sigma, \kappa}^0(\mathfrak{E})$ is bounded and if $f \in H_{\sigma, \kappa}^0(\mathfrak{E})$ then there is $C > 0$ such that $\|f(x, \zeta)\|_{\mathfrak{E}} \leq C \langle x, \zeta \rangle$ for any $(x, \zeta) \in \mathbb{R}^{2n}$. The following lemma will give us the behaviour of the $E_{\sigma, \kappa}$ and $H_{\sigma, \kappa}$ spaces under composition.

Lemma 5.4.22. (i) Let $f \in H_{\sigma, \kappa}^{w'}(\mathfrak{E})$ (resp. $E_{\sigma, \kappa}^{w'}(\mathfrak{E})$) and $g \in H_{\sigma, \kappa}^w(\mathbb{R}^{2n})$ such that there exists $C, c > 0$, $r \geq 0$, such that $\langle g_1(x, \zeta) \rangle \geq c \langle x \rangle \langle \zeta \rangle^{-r}$ (if $\sigma \neq 0$) and $\langle g_2(x, \zeta) \rangle \leq C \langle \zeta \rangle$ for any $(x, \zeta) \in \mathbb{R}^{2n}$, where $g = (g_1, g_2)$. Then $f \circ g \in H_{\sigma, \kappa+|w|+r\sigma}^{|w|+|w'|}(\mathfrak{E})$ (resp. $E_{\sigma, \kappa+|w|+r\sigma}^{|w'|}(\mathfrak{E})$).

(ii) If $P \in E_{\sigma, \kappa}^w(\mathcal{M}_n(\mathbb{R}))$, then $(x, \zeta) \mapsto P_{x, \zeta}(\zeta) \in H_{\sigma, \kappa}^{w+\kappa+1}(\mathbb{R}^n)$.

(iii) Let $f \in G_\sigma(\mathbb{R}^n, \mathfrak{E})$ and $g \in H_{\sigma, \kappa}^w(\mathbb{R}^n)$ such that there exists $c > 0$, $r \geq 0$, such that, if $\sigma \neq 0$, $\langle g(x, \zeta) \rangle \geq c \langle x \rangle \langle \zeta \rangle^{-r}$ for any $(x, \zeta) \in \mathbb{R}^{2n}$. Then $f \circ g \in H_{\sigma, \max\{r\sigma, \kappa\}+|w|}^{|w|}(\mathfrak{E})$. Moreover, if $f \in G_\sigma(\mathbb{R}^n, \mathbb{R}^p)$, then $df \circ g \in E_{\sigma, \max\{r\sigma, \kappa\}+|w|}^0(\mathcal{M}_{p, n}(\mathbb{R}))$.

Proof. (i) The Faa di Bruno formula yields for any $2n$ -multi-index $\nu \neq 0$,

$$\partial^\nu(f \circ g) = \sum_{1 \leq |\lambda| \leq |\nu|} (\partial^\lambda f) \circ g P_{\nu, \lambda}(g) \quad (5.28)$$

where $P_{\nu, \lambda}(g)$ is a linear combination (with coefficients independent of f and g) of functions of the form $\prod_{j=1}^s (\partial^{l^j} g)^{k^j}$ where $s \in \{1, \dots, |\nu|\}$. The k^j and l^j are $2n$ -multi-indices (for $1 \leq j \leq s$) such that $|k^j| > 0$, $|l^j| > 0$, $\sum_{j=1}^s k^j = \lambda$ and $\sum_{j=1}^s |k^j| l^j = \nu$. As a consequence, since $g \in H_{\sigma, \kappa}^w(\mathbb{R}^{2n})$, we see that for each ν, λ with $1 \leq |\lambda| \leq |\nu|$ there exists $C_{\nu, \lambda} > 0$ such that for any $(x, \zeta) \in \mathbb{R}^n$,

$$|P_{\nu, \lambda}(g)(x, \zeta)| \leq C_{\nu, \lambda} \langle x \rangle^{-\sigma(|\nu|-|\lambda|)} \langle \zeta \rangle^{w|\lambda|+\kappa(|\nu|-|\lambda|)}. \quad (5.29)$$

Moreover, since $f \in H_{\sigma, \kappa}^{w'}(\mathbb{R}^{2n})$ (resp. $E_{\sigma, \kappa}^{w'}(\mathbb{R}^{2n})$), there is $C'_\lambda > 0$ such that for any $(x, \zeta) \in \mathbb{R}^{2n}$, the estimate $\|(\partial^\lambda f) \circ g(x, \zeta)\| \leq C'_\lambda \langle x \rangle^{-\sigma(|\lambda|-1)} \langle \zeta \rangle^{|w'|+(\kappa+r\sigma)(|\lambda|-1)}$ (resp. $\|(\partial^\lambda f) \circ g(x, \zeta)\| \leq C'_\lambda \langle x \rangle^{-\sigma|\lambda|} \langle \zeta \rangle^{|w'|+(\kappa+r\sigma)|\lambda|}$) is valid. We deduce then from (5.28) and (5.29) that $f \circ g$ belongs to $H_{\sigma, \kappa+|w|+r\sigma}^{|w|+|w'|}(\mathfrak{E})$ (resp. $E_{\sigma, \kappa+|w|+r\sigma}^{|w'|}(\mathfrak{E})$).

(ii) We note $P_{x, \zeta}^{i, j}$ the matrix entries of $P_{x, \zeta}$. Each component $(f^i)_{1 \leq i \leq n}$ of the map $f := (x, \zeta) \mapsto P_{x, \zeta}(\zeta)$ is of the form $f^i = \sum_{j=1}^n P^{i, j} \zeta_j$. It is straightforward to check that the applications $(x, \zeta) \mapsto \zeta_j$ satisfy for any $\nu \in \mathbb{N}^{2n}$, $\partial^\nu \zeta_j = \mathcal{O}(\langle \zeta \rangle^{1-|\nu|} \langle x \rangle^{\sigma(1-|\nu|)})$. The result now follows from an application of the Leibniz rule.

(iii) Following the proof of (i), (5.29) is still valid, this time with λ as n -multi-indices and ν as $2n$ -multi-indices with $1 \leq |\lambda| \leq |\nu|$. Using the fact that $\langle g(x, \zeta) \rangle \geq c \langle x \rangle \langle \zeta \rangle^{-r}$ for any $(x, \zeta) \in \mathbb{R}^{2n}$, we obtain the following estimate

$$\|(\partial^\lambda f) \circ g(x, \zeta)\| \leq C'_\lambda \langle x \rangle^{-\sigma(|\lambda|-1)} \langle \zeta \rangle^{r\sigma(|\lambda|-1)} \leq C'_\lambda \langle x \rangle^{-\sigma(|\lambda|-1)} \langle \zeta \rangle^{\max\{r\sigma, \kappa\}(|\lambda|-1)}$$

which, with (5.29) and (5.28), yields $f \circ g$ belongs to $H_{\sigma, \max\{r\sigma, \kappa\}+|w|}^{|w|}(\mathfrak{E})$. The fact that $df \circ g$ is in $E_{\sigma, \max\{r\sigma, \kappa\}+|w|}^0(\mathcal{M}_{p,n}(\mathbb{R}))$ when $f \in G_\sigma(\mathbb{R}^n, \mathbb{R}^p)$ is based on the same argument. \square

The $H_{\sigma, \kappa}$ and $E_{\sigma, \kappa}$ spaces are related to the symbol and amplitude spaces by the following lemma.

Lemma 5.4.23. (i) If $f \in E_{\sigma, \kappa, z}^w$, then $(x, \zeta, \vartheta) \mapsto f(x, \zeta)$ is in $\Pi_{\sigma, \kappa, z}^{0, w, 0}$.

(ii) Let $s \in S_{\sigma, z}^{l, m}$, $m \in H_{\sigma, \kappa}^w(\mathbb{R}^n)$ such that there exist $C, c, r > 0$ such that, if $\sigma \neq 0$, for any $(x, \zeta) \in \mathbb{R}^{2n}$, $c \langle x \rangle \langle \zeta \rangle^{-r} \leq \langle m(x, \zeta) \rangle \leq C \langle x \rangle \langle \zeta \rangle^r$, and $P \in E_{\sigma, \kappa}^0(\mathcal{M}_n(\mathbb{R}))$ such that for any $(x, \zeta, \vartheta) \in \mathbb{R}^{3n}$, $\langle P_{x, \zeta}(\vartheta) \rangle \geq c \langle \vartheta \rangle$. Then $(x, \zeta, \vartheta) \mapsto s(m(x, \zeta), P_{x, \zeta}(\vartheta))$ is in $\Pi_{\sigma, \kappa+|\sigma r - \kappa + w|, z}^{l, \sigma r |l|, m}$.

(iii) If $s \in S_\sigma(\mathbb{R}^n)$, $m \in H_{\sigma, \kappa}^w(\mathbb{R}^n)$ such that, if $\sigma \neq 0$, there exists $c, r > 0$ such that for any $(x, \zeta) \in \mathbb{R}^{2n}$, $\langle m(x, \zeta) \rangle \geq c \langle x \rangle \langle \zeta \rangle^{-r}$, then $(x, \zeta, \vartheta) \mapsto s(m(x, \zeta)) \text{Id}_{L(E_z)}$ is in $\Pi_{\sigma, \kappa+|\sigma r - \kappa + w|, z}^{0, 0, 0}$.

(iv) If $a \in \Pi_{\sigma, \kappa, z}^{l, w, m}$ and $P \in E_{\sigma, \kappa}^0(\mathcal{M}_n(\mathbb{R}))$ is such that there is $c > 0$ such that for any $(x, \zeta, \vartheta) \in \mathbb{R}^{3n}$, $\langle P_{x, \zeta}(\vartheta) \rangle \geq c \langle \vartheta \rangle$, then $a_P : (x, \zeta, \vartheta) \mapsto a(x, \zeta, P_{x, \zeta}(\vartheta)) \in \Pi_{\sigma, \kappa, z}^{l, w, m}$.

Proof. (i) is straightforward.

(ii) Let us note $g(x, \zeta, \vartheta) := (m(x, \zeta), P_{x, \zeta}(\vartheta))$. For any $i, j \in \{1, \dots, n\}$, we denote $P_{x, \zeta}^{i, j}$ the (i, j) matrix entry of $P_{x, \zeta}$. Since $P \in E_{\sigma, \kappa}^0(\mathcal{M}_n(\mathbb{R}))$, we have $P_{\cdot, \cdot}^{i, j} \in E_{\sigma, \kappa}^0(\mathbb{R})$. Faà di Bruno formula in Theorem 5.2.11 yields for any $\nu \neq 0$

$$\partial^\nu (s \circ g) = \sum_{1 \leq |\lambda| \leq |\nu|} (P_{\nu, \lambda}(g)) (\partial^\lambda s) \circ g \quad (5.30)$$

where $P_{\nu, \lambda}(g)$ is a linear combination of terms of the form $\prod_{j=1}^s (\partial^{l^j} g)^{k^j}$, where $1 \leq s \leq |\nu|$, the k^j (resp. l^j) are $2n$ -multi-indices (resp. $3n$ -multi-indices) with $|k^j| > 0$, $|l^j| > 0$, $\sum_{j=1}^s k^j = \lambda$ and $\sum_{j=1}^s |k^j| l^j = \nu$. Let us note $l^j =: (l^{j,1}, l^{j,2}, l^{j,3})$, $k^j =: (k^{j,1}, k^{j,2})$ where $l^{j,1}, l^{j,2}, l^{j,3}, k^{j,1}, k^{j,2}$ are n -multi-indices. We have, noting $Q(x, \zeta, \vartheta) := (x, \zeta)$,

$$(\partial^{l^j} g)^{k^j} = \prod_{i=1}^n (\delta_{l^{j,3}, 0} (\partial^{(l^{j,1}, l^{j,2})} m)_i \circ Q)^{k_i^{j,1}} \prod_{i=1}^n \left(\sum_{k=1}^n \partial^{(l^{j,1}, l^{j,2})} P_{\cdot, \cdot}^{i, k} \partial^{l^{j,3}} \vartheta_k \right)^{k_i^{j,2}}$$

and we get, for a given s , (l^j) , (k^j) such that $(\partial^{l^j} g)^{k^j} \neq 0$ for all $1 \leq j \leq s$,

$$\begin{aligned} \text{if } |l^{j,3}| = 0, \quad & (\partial^{l^j} g)^{k^j} = \mathcal{O}(\langle x \rangle^{-\sigma|l^j| + \sigma|k^{j,1}|} \langle \zeta \rangle^{\kappa|l^j| + |k^{j,1}| + w|k^{j,1}|} \langle \vartheta \rangle^{|k^{j,2}|}), \\ \text{if } |l^{j,3}| = 1, \quad & k^{j,1} = 0 \text{ and } (\partial^{l^j} g)^{k^j} = \mathcal{O}(\langle x \rangle^{-\sigma|l^j| + \sigma|k^j|} \langle \zeta \rangle^{\kappa|l^j| + |k^j| - \kappa|k^j|}). \end{aligned}$$

The case $|l^{j,3}| > 1$ is excluded since $k^j \neq 0$ and $(\partial^{l^j} g)^{k^j} \neq 0$. By permutation on the j indices, we can suppose as in the proof of Lemma 5.2.13 that for $1 \leq j \leq j_1 - 1$, we have $l^{j,3} = 0$ and for

$j_1 \leq j \leq s$, we have $|l^{j,3}| = 1$, where $1 \leq j_1 \leq s+1$. Thus, we get

$$\prod_{j=1}^s (\partial^{l^j} g)^{k^j} = \mathcal{O}(\langle x \rangle^{-\sigma(\sum_{j=1}^s (|l^j|-1)|k^j| + \sum_{j=1}^{j_1-1} |k^{j,2}|)}) \\ \times \langle \zeta \rangle^{w \sum_{j=1}^s |k^{j,1}| + \kappa(\sum_{j=1}^s (|l^j|-1)|k^j| + \sum_{j=1}^{j_1-1} |k^{j,2}|)} \langle \vartheta \rangle^{\sum_{j=1}^{j_1-1} |k^{j,2}|}.$$

We check that $\sum_{j=1}^{j_1-1} |k^{j,2}| = |\lambda^2| - |\gamma|$ and $\sum_{j=1}^s (|l^j| - 1)|k^j| = |\nu| - |\lambda|$ where $\lambda = (\lambda^1, \lambda^2)$ and $\nu = (\alpha, \beta, \gamma)$. As a consequence,

$$P_{\nu, \lambda}(g) = \mathcal{O}(\langle x \rangle^{-\sigma(|\alpha+\beta|-|\lambda^1|)} \langle \zeta \rangle^{w|\lambda^1| + \kappa(|\alpha+\beta|-|\lambda^1|)} \langle \vartheta \rangle^{|\lambda^2|-|\gamma|}). \quad (5.31)$$

Since there exist $C, c > 0$ such that for any $(x, \zeta) \in \mathbb{R}^{2n}$ $\langle m(x, \zeta) \rangle \leq C \langle x \rangle \langle \zeta \rangle^r$ and $\langle m(x, \zeta) \rangle \geq c \langle x \rangle \langle \zeta \rangle^{-r}$, we see that there is $K_\nu > 0$ such that for any $1 \leq |\lambda| \leq |\nu|$ and any $(x, \zeta) \in \mathbb{R}^{2n}$, $\langle m(x, \zeta) \rangle^{\sigma(l-|\lambda^1|)} \leq K_\nu \langle x \rangle^{\sigma(l-|\lambda^1|)} \langle \zeta \rangle^{\sigma r |l| + \sigma r |\lambda^1|}$. As a consequence, we see that there is $C_\nu > 0$ such that for any $1 \leq |\lambda| \leq |\nu|$ and any $(x, \zeta, \vartheta) \in \mathbb{R}^{3n}$,

$$\left\| (\partial^\lambda s) \circ g(x, \zeta, \vartheta) \right\|_{L(E_z)} \leq C_\nu \langle x \rangle^{\sigma(l-|\lambda^1|)} \langle \zeta \rangle^{\sigma r |l| + \sigma r |\lambda^1|} \langle \vartheta \rangle^{m-|\lambda^2|}.$$

Thus, since we can reduce the sum in (5.30) to $2n$ -multi-indices λ such that $|\lambda^2| \geq |\gamma|$ (and thus $|\lambda^1| \leq |\alpha + \beta|$), we obtain the result from (5.31) and a straightforward verification of the case $\nu = 0$.

(iii) is obtain exactly as (ii) (with $P_{x, \zeta} = \text{Id}$), since $(x, \zeta) \mapsto \mu_{z, \mathbf{b}}(x) \text{Id}_{L(E_z)} \in S_{\sigma, z}^{0,0}$. The hypothesis $m(x, \zeta) = \mathcal{O}(\langle x \rangle \langle \zeta \rangle^r)$ is not necessary since $l = 0$ here.

(iv) We have, noting $g(x, \zeta, \vartheta) := (x, \zeta, P_{x, \zeta}(\vartheta))$, for any $3n$ -multi-indices $\nu \neq 0$, $1 \leq |\nu'| \leq |\nu|$, $P_{\nu, \nu'}(g)$ as a linear combination of terms of the form $\prod_{j=1}^s (\partial^{l^j} g)^{k^j}$, with $\sum_{j=1}^s |k^j| |l^j| = \nu$ and $\sum_{j=1}^s k^j = \nu'$, noting $k^j = (k^{j,1}, k^{j,2})$, $l^j = (l^{j,1}, l^{j,2})$, where $k^{j,1}$ and $l^{j,1}$ are $2n$ -multi-indices, we get, following the proof of (ii),

$$P_{\nu, \nu'}(g) = \mathcal{O}(\langle x \rangle^{-\sigma(|\alpha+\beta|-|\alpha'+\beta'|)} \langle \zeta \rangle^{\kappa(|\alpha+\beta|-|\alpha'+\beta'|)} \langle \vartheta \rangle^{|\gamma'|-|\gamma|}).$$

Since $P_{x, \zeta} = \mathcal{O}(1)$ and $\langle P_{x, \zeta}(\vartheta) \rangle \geq \varepsilon \langle \vartheta \rangle$ we get the result. \square

Definition 5.4.24. Let $\sigma \in [0, 1]$ and ψ a linearization on $(M, \exp, E, d\mu)$. We say that ψ is a S_σ -linearization if for any frame (z, \mathbf{b}) , there is $\kappa_{z, \mathbf{b}} \geq 0$ such that

- (i) $\psi_z^{\mathbf{b}} \in H_{\sigma, \kappa_{z, \mathbf{b}}}(\mathbb{R}^n)$ with $\psi_z^{\mathbf{b}}(x, \zeta) = \mathcal{O}(\langle x \rangle \langle \zeta \rangle^r)$ for a $r \geq 1$ and $\overline{\psi_z^{\mathbf{b}}} \in \mathcal{O}_M(\mathbb{R}^{2n}, \mathbb{R}^n)$,
- (ii) there is $P^{z, \mathbf{b}} \in C^\infty(\mathbb{R}^{2n}, GL_n(\mathbb{R}))$ such that $P^{z, \mathbf{b}}$ and $(P^{z, \mathbf{b}})^{-1}$ are in $E_{\sigma, \kappa_{z, \mathbf{b}}}^0(\mathcal{M}_n(\mathbb{R}))$, and for any $(x, \zeta) \in \mathbb{R}^{2n}$, $P_{x, \zeta}^{z, \mathbf{b}}(\zeta) = \Upsilon_{1, T}^{z, \mathbf{b}}(x, \zeta)$ and $P_{x, 0}^{z, \mathbf{b}} = \text{Id}_{\mathbb{R}^n}$.

- (iii) $\tau_1^{z, \mathbf{b}}$ and $(\tau_1^{z, \mathbf{b}})^{-1}$ are in $E_{\sigma, \kappa_{z, \mathbf{b}}}^0(L(E_z))$.

We shall say that the combo $(M, \exp, E, d\mu, \psi)$ has a S_σ -bounded geometry if this is the case of $(M, \exp, E, d\mu)$ and ψ is a S_σ -linearization.

It is clear that a S_σ -linearization is also a \mathcal{O}_M -linearization. Moreover, in case of S_σ -bounded geometry, we check the properties (i), (ii) and (iii) in just one frame:

Lemma 5.4.25. *If $(M, \exp, E, d\mu)$ has a S_σ -bounded geometry and ψ is a linearization such that there exists (z_0, \mathbf{b}_0) , $\kappa_{z_0, \mathbf{b}_0} \geq 0$, such that the functions $\psi_{z_0}^{\mathbf{b}_0}$, $\overline{\psi_{z_0}^{\mathbf{b}_0}}$ satisfy (i), (ii) and (iii), then ψ is a S_σ -linearization.*

Proof. This follows from applications of Lemma 5.4.22. \square

Remark 5.4.26. The condition (ii) in Definition 5.4.24 encodes an abstract parallel transport isomorphisms in normal coordinates. Indeed, in the case where the linearization ψ is derived from a connection on M , the $GL_n(\mathbb{R})$ -valued smooth functions on \mathbb{R}^{2n} : $P^{z,b} := (x, \zeta) \mapsto M_{z, \exp \circ (n_{z,T}^b)^{-1}(x, \zeta)} P_{(n_{z,T}^b)^{-1}(x, \zeta)} (M_{z, (n_z^b)^{-1}(x)}^{-1})^{-1}$ where the applications $P_{x, \xi}$ are the parallel transport isomorphisms on the tangent bundle (see Remark 5.3.3), satisfy for any $(x, \zeta) \in \mathbb{R}^{2n}$, $P_{x, \zeta}^{z,b}(\zeta) = \Upsilon_{1,T}^{z,b}(x, \zeta)$ and $P_{x,0}^{z,b} = \text{Id}_{\mathbb{R}^n}$. Thus, in this case, (ii) is satisfied if $P^{z,b}$ and $(P^{z,b})^{-1}$ are in $E_{\sigma, \kappa_{z,b}}^0(\mathcal{M}_n(\mathbb{R}))$ for a $\kappa_{z,b} \geq 0$.

Remark that for any $t \in \mathbb{R}$ and $(x, \zeta) \in \mathbb{R}^{2n}$, if $P^{z,b} \in C^\infty(\mathbb{R}^{2n}, GL_n(\mathbb{R}))$ satisfies (ii), then $P_{x, t\zeta}^{z,b}(\zeta) = \Upsilon_{t,T}^{z,b}(x, \zeta)$. We shall note $P_t^{z,b} := (x, \zeta) \mapsto P_{x, t\zeta}^{z,b}$, so that $P_1^{z,b} = P^{z,b}$ and $P_0^{z,b} = \text{Id}_{\mathbb{R}^n}$. Thus, $\Upsilon_{t,z,b}(x, \zeta) = (\psi_z^b(x, t\zeta), P_{t,x,\zeta}^{z,b}(\zeta))$ and we define the following diffeomorphism on \mathbb{R}^{3n} ,

$$\Xi_{t,z,b} := (x, \zeta, \vartheta) \mapsto (\Upsilon_{t,z,b}(x, \zeta), \tilde{P}_{t,x,\zeta}^{z,b}(\vartheta)). \quad (5.32)$$

We also define the \mathbb{R}^{2n} -valued function $\hat{\Xi}_{t,z,b} : (x, \zeta, \vartheta) \mapsto (\psi_z^b(x, t\zeta), \tilde{P}_{t,x,\zeta}^{z,b}(\vartheta))$. We check that $J(\Xi_{t,z,b}) = J(\Upsilon_{t,z,b}) (\det(P_t^{z,b})^{-1})$ and $J(\Xi_{t,z,b}^{-1}) = J(\Upsilon_{-t,z,b}) (\det(P_t^{z,b} \circ \Upsilon_{-t,z,b}))$. Note also that for any $(x, y) \in \mathbb{R}^{2n}$, $\overline{\psi_z^b(y, x)} = -P_{x, \psi_z^b(x, y)}^{z,b}(\psi_z^b(x, y))$.

Lemma 5.4.27. Let (z, b) be a given frame, $\lambda, \lambda' \in [0, 1]$ and $t \in [-1, 1]$. Suppose also that $(M, \exp, E, d\mu, \psi)$ has a S_σ -bounded geometry. Then

- (i) $P_t^{z,b}, (P_t^{z,b})^{-1}$ are in $E_{\sigma, \kappa_{z,b}}^0(\mathcal{M}_n(\mathbb{R}))$, and $\tau_t^{z,b}, (\tau_t^{z,b})^{-1}$ are in $E_{\sigma, \kappa_{z,b}}^0(L(E_z))$.
- (ii) $m_t^{z,b} := \psi_z^b \circ I_{1,t} \in H_{\sigma, \kappa_{z,b}}(\mathbb{R}^n)$ and there is $c > 0$, $r \geq 1$ such that for any $(x, \zeta) \in \mathbb{R}^{2n}$, $\langle m_t^{z,b}(x, \zeta) \rangle \geq c \langle x \rangle \langle \zeta \rangle^{-r}$.
- (iii) There is $c, \varepsilon > 0$ such that for any $(x, \zeta) \in \mathbb{R}^{2n}$, $\langle \psi_z^b(x, \zeta) \rangle \geq c \langle \zeta \rangle^\varepsilon \langle x \rangle^{-1}$.
- (iv) $\Phi_{\lambda,z,b} \in H_{\sigma, \kappa_{z,b}}(\mathbb{R}^{2n})$. In particular $J_{\lambda,z,b} \in E_{\sigma, \kappa_{z,b}}(\mathbb{R})$.
- (v) $\Upsilon_{t,z,b} \in H_{\sigma, \kappa_{z,b}}(\mathbb{R}^{2n})$. In particular $J(\Upsilon_{t,z,b}) \in E_{\sigma, \kappa_{z,b}}(\mathbb{R})$. Moreover, there is $C > 0$ such that $\langle \Upsilon_{t,T}^{z,b}(x, \zeta) \rangle \leq C \langle \zeta \rangle$ for any $(x, \zeta) \in \mathbb{R}^{2n}$.
- (vi) $J(\Xi_{t,z,b})$ and $J(\Xi_{t,z,b}^{-1})$ are in $E_{\sigma, \kappa_{z,b}}(\mathbb{R})$.

Proof. (i) The case $t = 0$ is obvious. Suppose $t \neq 0$. Since $P_t^{z,b} = P^{z,b} \circ I_{1,t}$ and $I_{1,t} \in H_{\sigma, \kappa_{z,b}}^0$ the result follows from Lemma 5.4.22 (i). The same argument is applied to $(P_t^{z,b})^{-1}$, $\tau_t^{z,b}$ and $(\tau_t^{z,b})^{-1}$.

(ii) We shall use the shorthand $m_t := m_t^{z,b}$. In the case $t = 0$, $m_0 = \pi_1$, so we obtain the result. Suppose $t \neq 0$. In that case Lemma 5.4.22 (i) entails that $m_t \in H_{\sigma, \kappa_{z,b}}(\mathbb{R}^n)$. Since $\Upsilon_{t,z,b} = (m_t, \Upsilon_{t,T}^{z,b})$, we see that $\langle \Upsilon_{t,z,b}(x, \zeta) \rangle = \mathcal{O}(\langle x \rangle \langle \zeta \rangle^r)$ for a $r \geq 1$. Thus, there is $C > 0$ such that for any $(x, \zeta) \in \mathbb{R}^{2n}$, we have $\langle m_t(x, \zeta) \rangle \langle P_{t,x,\zeta}^{z,b}(\zeta) \rangle^r \geq C \langle x, \zeta \rangle$. Since there is $K > 0$ such that for any $(x, \zeta) \in \mathbb{R}^{2n}$, $\langle P_{t,x,\zeta}^{z,b}(\zeta) \rangle \leq K \langle \zeta \rangle$, we obtain the desired estimate.

(iii) $V := (\pi_1, \psi_z^b)$ is a diffeomorphism on \mathbb{R}^{2n} with inverse $V^{-1} = (\pi_1, \overline{\psi_z^b})$. Since $\overline{\psi_z^b} = \mathcal{O}(\langle x, y \rangle^r)$ for a $r \geq 1$ by hypothesis, we see that there is $c > 0$ such that $\langle x, \psi_z^b(x, \zeta) \rangle \geq c \langle x, \zeta \rangle$ for any $(x, \zeta) \in \mathbb{R}^{2n}$. This yields the result.

(iv) Direct consequence of (ii) and the fact that $\Phi_{\lambda,z,b} = (m_\lambda, m_{\lambda-1})$.

(v) follows from a straightforward application of (ii), Lemma 5.4.22 (ii) and the fact that for any $(x, \zeta) \in \mathbb{R}^{2n}$, $\Upsilon_{t,z,b}(x, \zeta) = (m_t(x, \zeta), P_{t,x,\zeta}^{z,b}(\zeta))$.

(vi) By (i), (v) and Lemma 5.4.22 (i), $P_t^{z,b} \circ \Upsilon_{-t,z,b} \in E_{\sigma,\kappa}^0(\mathcal{M}_n(\mathbb{R}))$. Thus the result follows from (i), (v), and the formulas $J(\Xi_{t,z,b}) = J(\Upsilon_{t,z,b}) (\det(P_t^{z,b})^{-1})$ and $J(\Xi_{t,z,b}^{-1}) = J(\Upsilon_{-t,z,b}) (\det(P_t^{z,b} \circ \Upsilon_{-t,z,b}))$. \square

5.4.4 Pseudodifferential operators

Assumption 5.4.28. *We suppose in this section and until section 5.5 that $(M, \exp, E, d\mu, \psi)$ has a S_σ -bounded geometry.*

Definition 5.4.29. A pseudodifferential operator of order l, m and type σ is an element of $\Psi_\sigma^{l,m} := \mathfrak{Op}_\lambda(S_\sigma^{l,m})$, where $\lambda \in [0, 1]$.

By Lemma 5.4.7, $S_\sigma^{l,m}$ can be seen as included in $\mathcal{S}'(T^*M, L(E))$, so $\mathfrak{Op}_\lambda(S_\sigma^{l,m})$ is well defined. The following theorem shows that it does not depend on λ , and thus justify the notation $\Psi_\sigma^{l,m}$. We note $\tau_R^{\lambda,\lambda'} := (\tau_\lambda^{z,b})^{-1} \circ \Upsilon_{\lambda'-\lambda,z,b} \tau_{\lambda'}^{z,b}$ and $\tau_L^{\lambda,\lambda'} := (\tau_{\lambda'-1}^{z,b})^{-1} \tau_{\lambda-1}^{z,b} \circ \Upsilon_{\lambda'-\lambda}$. If $\psi = \exp$, we have $\tau_R^{\lambda,\lambda'} = \tau_{R,\lambda'-\lambda}$ and $\tau_L^{\lambda,\lambda'} = (\tau_{L,\lambda'-\lambda})^{-1}$ where $\tau_{L,t} := \tau_t^{z,b}$ if $t \neq 1$ and $\tau_{L,t} := (\tau_{-1}^{z,b})^{-1} \circ \Upsilon_{1,z,b}$ if $t = 1$, and $\tau_{R,t} := \tau_t^{z,b}$ if $t \neq -1$ and $\tau_{R,t} := (\tau_1^{z,b})^{-1} \circ \Upsilon_{-1,z,b}$ if $t = -1$.

Theorem 5.4.30. *Let $\lambda, \lambda' \in [0, 1]$ and $K = \mathfrak{Op}_\lambda(a)$, with $a \in S_\sigma^{l,m}$. Then there exists (an unique) $a' \in S_\sigma^{l,m}$ such that $K = \mathfrak{Op}_{\lambda'}(a')$. Moreover, for any frame (z, b) ,*

$$a'_{z,b} \sim \sum_{\beta} \frac{(i/2\pi)^{|\beta|}}{\beta!} (\partial^{(0,\beta,\beta)} \tau_L^{\lambda,\lambda'} a_{\lambda'-\lambda}^{z,b} \tau_R^{\lambda,\lambda'})_{\zeta=0}$$

where $a_{z,b} := T_{z,b,*}(a)$, $a'_{z,b} := T_{z,b,*}(a')$, and $a_t^{z,b}$ is the amplitude defined for any $t \in [-1, 1]$ as

$$a_t^{z,b}(x, \zeta, \vartheta) := \frac{\mu_{z,b}(m_t^{z,b}(x, \zeta))}{\mu_{z,b}(x)} |J\Xi_{t,z,b}(x, \zeta)| (a_{z,b} \circ \widehat{\Xi}_{t,z,b}(x, \zeta, \vartheta)).$$

Proof. Let us fix a frame (z, b) and note $a_{z,b} := T_{z,b,*}(a)$. We saw in Remark 5.4.16 that $\mathfrak{Op}_\lambda(a)_{z,b} = \mathfrak{Op}_{\Gamma_{\lambda,z,b}}(\mu a_{z,b})$. Thus, for any $u \in \mathcal{S}(M \times M, L(E))$, we have with $u_{z,b} := T_{z,b,M^2}(u) \in \mathcal{S}(\mathbb{R}^{2n}, L(E_z))$,

$$\langle K, u \rangle = \int_{\mathbb{R}^{3n}} e^{2\pi i \langle \vartheta, \zeta \rangle} \text{Tr} (\mu a_{z,b}(x, \vartheta) (\Gamma_{\lambda,z,b}(u_{z,b})(x, \zeta))^*) d\zeta d\vartheta dx.$$

Suppose that $m \leq -2n$ so that the integral is absolutely convergent. We now proceed to the global change of variables provided by the diffeomorphism $\Xi_{\lambda'-\lambda}^{z,b}$ of \mathbb{R}^{3n} ($\Xi_{t,z,b}$ is defined at (5.32)). We get $\langle K, u \rangle = \langle \mathfrak{Op}_{\lambda',z,b}(\mu \tau_L^{\lambda,\lambda'} a_{\lambda'-\lambda}^{z,b} \tau_R^{\lambda,\lambda'}), u_{z,b} \rangle$. We check with Lemmas 5.4.27 and 5.4.23 that $\tau_L^{\lambda,\lambda'} a_{\lambda'-\lambda}^{z,b} \tau_R^{\lambda,\lambda'}$ is an amplitude in $\Pi_{\sigma,\kappa,z}^{l,w,m}$ for a $\kappa \geq 0$ and a $w \in \mathbb{R}$. We also see that the linear map $a_{z,b} \mapsto \mu \tau_L^{\lambda,\lambda'} a_{\lambda'-\lambda}^{z,b} \tau_R^{\lambda,\lambda'}$ is continuous on $S_{\sigma,z}^{l,m}$, which yields, using Proposition 5.4.14 (ii) and the density result of Lemma 5.4.6, the equality $\langle K, u \rangle = \langle \mathfrak{Op}_{\lambda',z,b}(\mu \tau_L^{\lambda,\lambda'} a_{\lambda'-\lambda}^{z,b} \tau_R^{\lambda,\lambda'}), u_{z,b} \rangle$, for any order m of the symbol a . The result now follows from Lemma 5.4.21 (iii). \square

Proposition 5.4.31. *For each $\lambda \in [0, 1]$ and $l, m \in \mathbb{R}$, σ_λ is a linear isomorphism from $\Psi_\sigma^{l,m}$ onto $S_\sigma^{l,m}$ and $\sigma_\lambda(A^\dagger) = (\sigma_{1-\lambda}(A))^*$ for any $A \in \Psi_\sigma^{l,m}$. In particular a pseudodifferential operator A is formally selfadjoint (i.e $A = A^\dagger$ as operators on \mathcal{S}) if and only if its Weyl symbol $\sigma_W(A)$ is selfadjoint (as a $L(E) \rightarrow T^*M$ section).*

Proof. The fact that σ_λ is a linear isomorphism from $\Psi_\sigma^{l,m}$ onto $S_\sigma^{l,m}$ is a consequence Theorem 5.4.30 and the fact that σ_λ is a topological isomorphism from $\mathcal{S}'(M \times M, L(E))$ onto $\mathcal{S}'(T^*M, L(E))$. We check that for any $T \in \mathcal{S}'(T^*M, L(E))$, $\mathfrak{Op}_\lambda(T)^\dagger = \mathfrak{Op}_{1-\lambda}(T^*)$ which is a direct consequence of the fact that $\Phi_\lambda(x, -\xi) = j \circ \Phi_{1-\lambda}(x, \xi)$ where $j(x, y) = (y, x)$. \square

Proposition 5.4.32. *Any operator in $\Psi_\sigma^{l,m}$ is regular. Moreover, for any $A \in \Psi_\sigma^{l,m}$ and $v \in \mathcal{S}$, we have*

$$A(v) : x \mapsto \int_{T_x^*(M)} d\mu_x^*(\theta) \int_{T_x(M)} d\mu_x(\xi) e^{2\pi i \langle \theta, \xi \rangle} \sigma_0(A)(x, \theta) \tau_{-1}^{-1}(x, \xi) v(\psi_x^{-\xi}).$$

Proof. Let $A \in \Psi_\sigma^{l,m}$ and $a := \sigma_0(A)$. Thus, for any frame (z, \mathbf{b}) , $A_{z,\mathbf{b}} = \mathfrak{Op}_{\Gamma_{0,z,\mathbf{b}}}(\mu a_{z,\mathbf{b}})$ so by Lemmas 5.4.17, 5.4.27 (ii) and (iii), $A_{z,\mathbf{b}}$ is continuous from $\mathcal{S}(\mathbb{R}^n, E_z)$ into itself. By Proposition 5.4.31, A^\dagger is a pseudodifferential operator in $\Psi_\sigma^{l,m}$, so we also obtain $(A^\dagger)_{z,\mathbf{b}}$ continuous from $\mathcal{S}(\mathbb{R}^n, E_z)$ into itself. The result follows. \square

5.4.5 Link with standard pseudodifferential calculus on \mathbb{R}^n and L^2 -continuity

We suppose in this section that E is the scalar bundle. If $A \in \Psi_\sigma$, then $A_{z,\mathbf{b}}$ belongs to the space, noted $\Psi_{\sigma,\psi}$, of regular operators B on $\mathcal{S}(\mathbb{R}^n)$, of the form

$$B(v)(x) = \int_{\mathbb{R}^{2n}} e^{2\pi i \langle \vartheta, \zeta \rangle} a(x, \vartheta) v(\psi_z^{\mathbf{b}}(x, -\zeta)) d\zeta d\vartheta$$

where $a \in S_\sigma^\infty(\mathbb{R}^{2n})$. We study in this section a sufficient condition on ψ , such that this space $\Psi_{\sigma,\psi}$ is in fact equal to the usual algebra $\Psi_{\sigma,std}$ pseudodifferential operators on \mathbb{R}^n with the standard linearization $\psi(x, \zeta) = x + \zeta$. Here $\Psi_{0,std}$ corresponds to the Hörmander calculus [81] on \mathbb{R}^n and $\Psi_{1,std}$ is the SG -calculus on \mathbb{R}^n .

We will note $\psi := \psi_z^{\mathbf{b}}$, $V_x(\zeta) := -\psi(x, -\zeta) + x$, $M_{x,\zeta} := [\int_0^1 \partial_j (V_x^{-1})^i(t\zeta) dt]_{i,j}$ and $N_{x,\zeta} := [\int_0^1 \partial_j V_x^i(t\zeta) dt]_{i,j}$. We consider the following hypothesis, noted (H_V) :

- (i) there is $\varepsilon, \delta, \eta > 0$ such that for any $(x, \zeta) \in \mathbb{R}^{2n}$ with $\|\zeta\| \leq \varepsilon \langle x \rangle^{\sigma\eta}$, we have $\det M_{x,\zeta} \geq \delta$ and $\det N_{x,\zeta} \geq \delta$,
- (ii) the functions $(dV_x)_{x,\zeta}$ and $(dV_x^{-1})_{x,\zeta}$ are in $E_\sigma^0(\mathcal{M}_n(\mathbb{R}))$.

Proposition 5.4.33. *If the hypothesis (H_V) holds, we have $\Psi_{\sigma,\psi} = \Psi_{\sigma,std}$.*

We set $\chi_{\varepsilon,\eta}(x, \zeta) := b(\frac{\|\zeta\|^2}{\varepsilon^2 \langle x \rangle^{2\sigma\eta}})$ where $b \in C_c^\infty(\mathbb{R}, [0, 1])$ is such that $b = 0$ on $\mathbb{R} \setminus]-1, 1[$ and $b = 1$ on $[-1/4, 1/4]$.

Lemma 5.4.34. *Suppose (H_V) . If $a \in S_\sigma^{l,m}(\mathbb{R}^{2n})$, then the application*

$$a_{\chi,M} : (x, \zeta, \vartheta) \mapsto \chi_{\varepsilon,\eta}(x, \zeta) a(x, \widetilde{M_{x,\zeta}} \vartheta) |J(V_x^{-1})|(\zeta) (\det M_{x,\zeta})^{-1}$$

is an amplitude in $\cup_{k,w} \Pi_{\sigma,\kappa,z}^{l,w,m}(\mathbb{R}^{3n})$. Similarly,

$$a_{\chi,N} : (x, \zeta, \vartheta) \mapsto \chi_{\varepsilon,\eta}(x, \zeta) a(x, \tilde{N}_{x,\zeta} \vartheta) |J(V_x)|(\zeta) (\det N_{x,\zeta})^{-1}$$

is in $\cup_{k,w} \Pi_{\sigma,\kappa,z}^{l,w,m}(\mathbb{R}^{3n})$.

Proof. The result follows from Lemma 5.4.23 (ii) and applications of Proposition 5.5.4. \square

Proof of Proposition 5.4.33. Suppose that $a \in S_{\sigma}^{l,m}(\mathbb{R}^{2n})$ and define A as the operator in $\Psi_{\sigma,\psi}$ with normal symbol a . We obtain for any $v \in \mathcal{S}(\mathbb{R}^{2n})$

$$A(v)(x) := \int_{\mathbb{R}^{2n}} e^{2\pi i \langle \vartheta, \zeta \rangle} a(x, \vartheta) v(\psi(x, -\zeta)) d\zeta d\vartheta.$$

We suppose first that $a \in S_{\sigma}^{-\infty}(\mathbb{R}^{2n})$. We have after a change of variable, and cutting the integral in two parts $A(v)(x) = A_1(v)(x) + A_2(v)(x)$ where

$$\begin{aligned} A_1(v)(x) &= \int_{\mathbb{R}^{2n}} e^{2\pi i \langle \vartheta, M_{x,\zeta}(\zeta) \rangle} \chi_{\varepsilon,\eta}(x, \zeta) a(x, \vartheta) |J(V_x^{-1})|(\zeta) v(x - \zeta) d\zeta d\vartheta, \\ A_2(v)(x) &= \int_{\mathbb{R}^{2n}} e^{2\pi i \langle \vartheta, V_x^{-1}(\zeta) \rangle} (1 - \chi_{\varepsilon,\eta})(x, \zeta) a(x, \vartheta) |J(V_x^{-1})|(\zeta) v(x - \zeta) d\zeta d\vartheta. \end{aligned}$$

In A_1 , we permute the integrations $d\zeta$ and $d\vartheta$ and proceed to a change of the variable ϑ , while in A_2 we integrate by parts in ϑ using formula (5.26) so that for any $p \in \mathbb{N}$,

$$\begin{aligned} A_1(v)(x) &= \int_{\mathbb{R}^{2n}} e^{2\pi i \langle \vartheta, \zeta \rangle} a_{\chi,M}(x, \zeta, \vartheta) v(x - \zeta) d\zeta d\vartheta, \\ A_2(v)(x) &= \int_{\mathbb{R}^{2n}} e^{2\pi i \langle \vartheta, V_x^{-1}(\zeta) \rangle} (1 - \chi_{\varepsilon,\eta})(x, \zeta) {}^t M_{\vartheta}^{p, V_x^{-1}(\zeta)}(a) |J(V_x^{-1})|(\zeta) v(x - \zeta) d\zeta d\vartheta. \end{aligned}$$

As a consequence with Lemma 5.4.34, and with the density of $S_{\sigma}^{-\infty}(\mathbb{R}^{2n})$ in $S_{\sigma}^{l,m}(\mathbb{R}^{2n})$, we see that A is the sum of two pseudodifferential operators in $\Psi_{\sigma,std}$: $A = A_{\chi} + R$ where $R \in \Psi_{\sigma,std}^{-\infty}$ and A_{χ} has $a_{\chi,M}$ as (standard) amplitude. The implication in the other sense is similar. \square

Remark 5.4.35. In the case of pseudodifferential operator with local compact control over the x variable and with ψ coming from a connection, by cutting-off in the ζ -variable or in other words taking $y := \psi(x, -\zeta)$ and x sufficiently close to each other, we have in fact $\Psi_{\sigma,\psi}$ equal to $\Psi_{\sigma,std}$ modulo smoothing elements (see [125]).

As a consequence, we see that if the hypothesis (H_V) is satisfied for a frame (z, \mathbf{b}) , then $\Psi_{\sigma,\psi}(= \Psi_{\sigma,std})$ is stable under composition of operators and the symbol composition formula is then given by a quadruple asymptotic summation modulo smoothing symbols.

We will show in the next section that we can also obtain stability under composition directly, without using a reduction to the standard calculus on \mathbb{R}^n . We shall obtain with this method a simpler symbol composition formula on $\Psi_{\sigma,\psi}$, analog to the usual one on $\Psi_{\sigma,std}$.

As a direct consequence of the previous proposition, we have the following L^2 -continuity result for pseudodifferential operators on M .

Proposition 5.4.36. If (H_V) is satisfied for the function V_x^{-1} in a frame (z, \mathbf{b}) , then any pseudodifferential operators on M of order $(0, 0)$ extends as a bounded operator on $L^2(M, d\mu)$.

Proof. Since (H_V) is satisfied for V_x^{-1} , the proof of the previous proposition entails that $\Psi_{\sigma,\psi}^{0,0} \subseteq \Psi_{\sigma,std}^{0,0}$, so the result follows from the L^2 -continuity of standard pseudodifferential operators [81]. \square

5.4.6 Composition of pseudodifferential operators

The goal of this section is to prove that pseudodifferential operators of Ψ_σ^∞ are stable under composition without using the hypothesis of the previous section, and to obtain an adapted symbol composition formula. We shall adapt to our situation a technique used for Fourier integral operators in Coriasco [42], Ruzhansky and Sugimoto [117, 119].

Let us note for $(x, \xi) \in TM$ and $\xi' \in T_{\psi_x^{-\xi}}(M)$, $\psi_{x, \xi, \xi'} := \psi_{\psi_x^{-\xi}}^{-\xi'}$, $r_x(\xi, \xi') := \psi_x^{-1}(\psi_{x, \xi, \xi'})$ and $q_x(\xi, \xi') := \psi_{\psi_x^{-\xi}, \xi'}^{-1}(\psi_x^{-\xi})$. We define V_x the $2n$ dimensional smooth manifold as $V_x := \{(\xi, \xi') \in T_x(M) \times \cup_{y \in M} T_y(M) \mid \xi' \in T_{\psi_x^{-\xi}}(M)\}$. Each V_x manifold is diffeomorphic to \mathbb{R}^{2n} via the map, defined for any fixed frame (z, \mathbf{b}) , $n_{z, V_x}^{\mathbf{b}}(\xi, \xi') := (M_{z, x}^{\mathbf{b}}(\xi), M_{z, \psi_x^{-\xi}}^{\mathbf{b}}(\xi'))$, and has a canonical involutive diffeomorphism R_x defined as

$$R_x : (\xi, \xi') \mapsto (r_x(\xi, \xi'), q_x(\xi, \xi')).$$

In all the following we fix a frame (z, \mathbf{b}) , and note also ψ the function $m_{-1}^{z, \mathbf{b}}$. We note $x^{\zeta, \zeta'} := \psi(\psi(x, \zeta), \zeta')$. For each $x \in \mathbb{R}^n$, $R_x := n_{z, V_{(n_z^{\mathbf{b}})^{-1}(x)}}^{\mathbf{b}} \circ R_{(n_z^{\mathbf{b}})^{-1}(x)} \circ (n_{z, V_{(n_z^{\mathbf{b}})^{-1}(x)}}^{\mathbf{b}})^{-1}$ is a diffeomorphism on \mathbb{R}^{2n} , and we define $R_x := (r_x, q_x)$, $r = r^{z, \mathbf{b}} := (x, \zeta, \zeta') \mapsto r_x(\zeta, \zeta')$ and $q = q^{z, \mathbf{b}} := (x, \zeta, \zeta') \mapsto q_x(\zeta, \zeta')$. Remark that $r_x(\zeta, \zeta') = -\psi_z^{\mathbf{b}}(x, x^{\zeta, \zeta'}) =: \bar{\psi}_x \circ \psi_{\psi_x(\zeta)}(\zeta')$ and $q_x(\zeta, \zeta') = -P_{-1, \psi(x, \zeta), \zeta'}^{z, \mathbf{b}}(\zeta')$. The map $r_{x, \zeta} : \zeta' \mapsto r_x(\zeta, \zeta')$ is a diffeomorphism on \mathbb{R}^n such that $r_{x, \zeta}^{-1} = r_{\psi_x(\zeta), \bar{\psi}_{\psi(x, \zeta)}(x)}$ so that $(dr_{x, \zeta})_{\zeta'}^{-1} = (dr_{\psi_x(\zeta), \bar{\psi}_{\psi(x, \zeta)}(x)})_{r_{x, \zeta}(\zeta')}$. We will use the shorthand $\tau := (\tau_{-1}^{z, \mathbf{b}})^{-1}$.

We note $s(x, \zeta, \zeta') := r(x, \zeta, \zeta') - \zeta$. We have $s(x, \zeta, \zeta') = s_{x, \zeta}(\zeta')$ where $s_{x, \zeta} = T_{-\zeta} \circ \bar{\psi}_x \circ \psi_{\psi_x(\zeta)}$ is a diffeomorphism on \mathbb{R}^n such that $s_{x, \zeta}(0) = 0$. We also define

$$\varphi_{x, \zeta}(\zeta') := r_{x, \zeta}(\zeta') - \zeta - (dr_{x, \zeta})_0(\zeta')$$

so that $\varphi_{x, \zeta}(0) = 0$ and $(d\varphi_{x, \zeta})_0 = 0$, and

$$V(x, \zeta, \zeta') := (dr_{x, \zeta})_{\zeta'}$$

as a smooth function from \mathbb{R}^{3n} into $\mathcal{M}_n(\mathbb{R})$. We shall note $(x, \zeta) \mapsto L_{x, \zeta} := -{}^t(dr_{x, \zeta})_0$.

We define $\mathcal{O}_{\sigma, \kappa, \varepsilon_0, \varepsilon_1, c}^{l, w_0, w_1}(\mathfrak{E})$, where $c \in \mathbb{N}$, $l \in \mathbb{R}$, $w := (w_0, w_1) \in \mathbb{R}_+^2$, $\varepsilon := (\varepsilon_0, \varepsilon_1)$, $\varepsilon_0 \geq 0$, $\varepsilon_1 > 0$, $\sigma \in [0, 1]$ and $\kappa \geq 0$, as the space of smooth functions g from \mathbb{R}^{3n} into \mathfrak{E} such that for any $3n$ -multi-index $\nu = (\mu, \gamma) \in \mathbb{N}^{2n} \times \mathbb{N}^n$, there exists $C_\nu > 0$ such that for any $(x, \zeta, \zeta') \in \mathbb{R}^{3n}$, $\|\partial^\nu g(x, \zeta, \zeta')\| \leq C_\nu \langle x \rangle^{\sigma(l - |\mu| - \varepsilon_1 |\gamma|_c)} \langle \zeta \rangle^{w_0 + \kappa |\mu| + \varepsilon_0 |\gamma|} \langle \zeta' \rangle^{w_1 + \kappa |\nu|}$. Here, we denoted $|\gamma|_c := 0$ if $|\gamma| < c$ and $|\gamma|_c := |\gamma| - c$ if $|\gamma| \geq c$. We note $\mathcal{O}_{\sigma, \kappa, \varepsilon}(\mathfrak{E}) := \cup_{c, l, w} \mathcal{O}_{\sigma, \kappa, \varepsilon, c}^{l, w}(\mathfrak{E})$. We check that for any multi-indices γ, γ' and $c, c' \in \mathbb{N}$, $|\gamma|_c + |\gamma'|_c \geq |\gamma + \gamma'|_{c+c'}$, and $|\gamma + \gamma'|_c \geq |\gamma|_c + |\gamma'|_c$. Thus, $\mathcal{O}_{\sigma, \kappa, \varepsilon}(\mathbb{R})$, $\mathcal{O}_{\sigma, \kappa, \varepsilon}(\mathcal{M}_p(\mathbb{R}))$ and $\mathcal{O}_{\sigma, \kappa, \varepsilon, z} := \mathcal{O}_{\sigma, \kappa, \varepsilon}(L(E_z))$ are algebras (graduated by the parameters c, l, w_0 and w_1) and $\partial^\nu \mathcal{O}_{\sigma, \kappa, \varepsilon, c}^{l, w}(\mathfrak{E}) \subseteq \mathcal{O}_{\sigma, \kappa, \varepsilon}^{l - |\mu| - \varepsilon_1 |\gamma|_c, w_0 + \kappa |\mu| + \varepsilon_0 |\gamma|, w_1 + \kappa |\nu|}(\mathfrak{E})$. If $f \in \mathcal{O}_{\sigma, \kappa, \varepsilon, c}^{0, w}(\mathfrak{E})$, then $(x, \zeta) \mapsto f(x, \zeta, 0) \in E_{\sigma, \kappa}^{w_0}(\mathfrak{E})$, and if $f \in \mathcal{O}_{\sigma, \kappa, \varepsilon, c, z}^{l, w}$, then $(x, \zeta, \vartheta) \mapsto f(x, \zeta, 0) \in \Pi_{\sigma, \kappa, z}^{l, w_0, 0}$. Remark that any monomial of the form $(x, \zeta, \zeta') \mapsto \zeta'^{\beta}$ where $\beta \in \mathbb{N}^n$, is in $\mathcal{O}_{\sigma, \kappa, \varepsilon, |\beta|}^{0, 0, |\beta|}(\mathbb{R})$ for any $\kappa \geq 0$ and $\varepsilon_0 \geq 0, \varepsilon_1 > 0$.

In the definition of S'_σ bounded geometry, we only require a polynomial control over the $\bar{\psi}_z^{\mathbf{b}}$ functions. It appears that for the theorem of composition, a stronger control over these functions is important. We thus introduce the following:

Definition 5.4.37. We shall say that (C_σ) is satisfied if there is a frame (z, \mathbf{b}) , $(\kappa_v, w_v) \in \mathbb{R}_+^2$ with $\kappa_v \geq 1$, and $\varepsilon_v \in]0, 1[$, such that

$$V \in \mathcal{O}_{\sigma, \kappa_v, \varepsilon_v, \varepsilon_v, 0}^{0,0,w_v}(\mathcal{M}_n(\mathbb{R})), \quad \text{and} \quad (d\psi_{z,x}^{\mathbf{b}})_\zeta, (d\bar{\psi}_{z,x}^{\mathbf{b}})_y = \mathcal{O}(1). \quad (5.33)$$

In particular (C_σ) entails that $(dr_{x,\zeta})_0$ and thus L are in $E_{\sigma, \kappa_v}^0(\mathcal{M}_n(\mathbb{R}))$.

We note $\mathcal{R}_{\sigma, \kappa, \varepsilon_1}^{w_0, w_1}(\mathfrak{E})$ ($\varepsilon_1 > 0$) as the space of smooth functions g such that for any nonzero $\nu = (\mu, \gamma) \in \mathbb{N}^{2n} \times \mathbb{N}^n$, $\partial^\nu g = \mathcal{O}(\langle x \rangle^{\sigma(1-|\mu|-\varepsilon_1|\gamma|)} \langle \zeta \rangle^{w_0+\kappa(|\nu|-1)} \langle \zeta' \rangle^{w_1+\kappa(|\nu|-1)})$. It follows from (C_σ) that $r \in \cup_{w_0, w_1} \mathcal{R}_{\sigma, \kappa_v, \varepsilon_v/2}^{w_0, w_1}(\mathbb{R}^n)$.

The following lemma will give us the link between the \mathcal{O} , \mathcal{R} , H , E spaces and the behaviour under composition.

Lemma 5.4.38. (i) Let $f \in H_{\sigma, \kappa}^w(\mathfrak{E})$ (resp. $E_{\sigma, \kappa}^w(\mathfrak{E})$) and $g \in \mathcal{R}_{\sigma, \kappa, \varepsilon_1}^{w_0, w_1}(\mathbb{R}^{2n})$ such that $g_2(x, \zeta, \zeta') = \mathcal{O}(\langle \zeta \rangle^{k_2} \langle \zeta' \rangle^{k'_2})$ for a $(k_2, k'_2) \in \mathbb{R}_+^2$ and, if $\sigma \neq 0$, $\langle g_1(x, \zeta, \zeta') \rangle \geq c \langle x \rangle \langle \zeta \rangle^{-k_1} \langle \zeta' \rangle^{-k'_1}$, for a $(k_1, k'_1) \in \mathbb{R}_+^2$ and $c > 0$. Then, $f \circ g \in \mathcal{R}_{\sigma, \kappa_H, \varepsilon_1}^{w_0+k_2w, w_1+k'_2w}(\mathfrak{E})$ (resp. $\mathcal{O}_{\sigma, \kappa_E, \kappa_E, \varepsilon_1, 0}^{0, k_2w, k'_2w}(\mathfrak{E})$) where $\kappa_H := \kappa + \max\{|w_0+k_1\sigma+k_2\kappa|, |w_1+k'_1\sigma+k'_2\kappa|\}$ and $\kappa_E := \kappa + \max\{|w_0+k_1\sigma+(k_2-1)\kappa|, |w_1+k'_1\sigma+(k'_2-1)\kappa|\}$.

(ii) $(x, \zeta, \zeta') \mapsto (\psi(x, \zeta), \zeta') \in \mathcal{R}_{\sigma, \kappa_\psi, 1}^{w_\psi, 0}(\mathbb{R}^{2n})$ and $(x, \zeta, \zeta') \mapsto x^{\zeta, \zeta'} \in \mathcal{R}_{\sigma, \kappa_\psi, 1}$ for a $(\kappa_\psi, w_\psi) \in \mathbb{R}_+^2$.

(iii) The functions $q, (x, \zeta, \zeta') \mapsto (P_{-1, \psi(x, \zeta), \zeta'}^{z, \mathbf{b}})^{-1}$ and $(x, \zeta, \zeta') \mapsto \det(P_{-1, \psi(x, \zeta), \zeta'}^{z, \mathbf{b}})^{-1}$ are respectively in $\mathcal{R}_{\sigma, \kappa_q, 1}(\mathbb{R}^n)$, $\mathcal{O}_{\sigma, \kappa_q, \kappa_q, 1, 0}^{0,0,0}(\mathcal{M}_n(\mathbb{R}))$, and $\mathcal{O}_{\sigma, \kappa_q, \kappa_q, 1, 0}^{0,0,0}(\mathbb{R})$, for a $\kappa_q \geq 0$. Moreover, there exists $C > 0$ such that for any $(x, \zeta, \zeta') \in \mathbb{R}^{3n}$, $\|q_x(\zeta, \zeta')\| \leq C \langle \zeta' \rangle$.

(iv) $(x, \zeta, \zeta') \mapsto \tau(x^{\zeta, \zeta'}, q_x(\zeta, \zeta'))$ is in $\mathcal{O}_{\sigma, \kappa_\tau, \kappa_\tau, 1, 0, z}^{0,0,0}$ for a $\kappa_\tau \geq 0$.

Proof. (i) If $\nu = (\alpha, \beta, \gamma) \neq 0$ is a $3n$ -multi-index, we have $\partial^\nu f \circ g = \sum_{1 \leq |\nu'| \leq |\nu|} P_{\nu, \nu'}(g)(\partial^{\nu'} f) \circ g$, with $P_{\nu, \nu'}(g)$ a linear combination of terms of the form $\prod_{j=1}^s (\partial^{l_j} g)^{k_j}$, with $1 \leq s \leq |\nu|$, $\sum_1^s l^j |k^j| = \nu$, $\sum_1^s k^j = \nu'$. As a consequence, we get the following estimate for any $1 \leq |\nu| \leq |\nu'|$, $P_{\nu, \nu'}(g) = \mathcal{O}(\langle x \rangle^{\sigma(|\nu'|-|\mu|-\varepsilon_1|\gamma|)} \langle \zeta \rangle^{w_0|\nu'|+\kappa(|\nu|-|\nu'|)} \langle \zeta' \rangle^{w_1|\nu'|+\kappa(|\nu|-|\nu'|)})$. Moreover, for any $1 \leq |\nu'| \leq |\nu|$, there is $C_\nu > 0$ such that for any $(x, \zeta, \zeta') \in \mathbb{R}^{3n}$, the following estimate is valid $\|(\partial^{\nu'} f) \circ g(x, \zeta, \zeta')\| \leq C_\nu \langle x \rangle^{-\sigma(|\nu'|-1)} \langle \zeta \rangle^{(k_1\sigma+k_2\kappa)(|\nu'|-1)+k_2w} \langle \zeta' \rangle^{(k'_1\sigma+k'_2\kappa)(|\nu'|-1)+k'_2w}$ (resp. $\|(\partial^{\nu'} f) \circ g(x, \zeta, \zeta')\| \leq C_\nu \langle x \rangle^{-\sigma|\nu'|} \langle \zeta \rangle^{(k_1\sigma+k_2\kappa)|\nu'|+k_2w} \langle \zeta' \rangle^{(k'_1\sigma+k'_2\kappa)|\nu'|+k'_2w}$). The result follows.

(ii) By hypothesis, $\psi \in H_{\sigma, \kappa_\psi}^{w_\psi, 0}$. We deduce that $(x, \zeta, \zeta') \mapsto \psi(x, \zeta) \in \mathcal{R}_{\sigma, \kappa_\psi, 1}^{w_\psi, 0}$ and the first statement now follows from $(x, \zeta, \zeta') \mapsto \zeta' \in \mathcal{R}_{\sigma, \kappa_\psi, 1}^{0,0}$. The second statement follows from (i).

(iii) Since $q_x(\zeta, \zeta') = -P_{-1, \psi(x, \zeta), \zeta'}^{z, \mathbf{b}}(\zeta')$, the fact that $q_x \in \mathcal{R}_{\sigma, \kappa_q, 1}(\mathbb{R}^n)$ for a $\kappa_q \geq 0$ is a consequence of (i), (ii) and Lemma 5.4.22 (iii). We also have by (i) and (ii), $(P_{-1, \psi(x, \zeta), \zeta'}^{z, \mathbf{b}})^{-1} \in \mathcal{O}_{\sigma, \kappa_q, \kappa_q, 1, 0}^{0,0,0}(\mathcal{M}_n(\mathbb{R}))$.

(iv) Since $\tau \in E_{\sigma, \kappa}^0(L(E_z))$ for a $\kappa \geq 0$, the result follows (i), (ii), (iii) and the estimate $\langle x^{\zeta, \zeta'} \rangle \geq c \langle x \rangle \langle \zeta \rangle^{-k} \langle \zeta' \rangle^{-k}$ for $c, k > 0$. \square

Lemma 5.4.39. Suppose (C_σ) . Then

(i) $s, \varphi \in \mathcal{O}_{\sigma, \kappa_v, \varepsilon_v, \varepsilon_v, 1}^{0,0,w_s}(\mathbb{R}^n)$ and $\varphi \in \mathcal{O}_{\sigma, \kappa_v, \varepsilon_v, \varepsilon_v, 2}^{-\varepsilon_v, \varepsilon_v, w_\varphi}(\mathbb{R}^n)$ where $w_s := w_v + 1$ and $w_\varphi := 2 + w_v + \kappa_v$.

(ii) $V = (dr_{x, \zeta})_{\zeta'}^{-1}$ and $(dr_{x, \zeta})_{\zeta'}^{-1}$ are bounded on \mathbb{R}^{3n} .

(iii) The function $J(R) : (x, \zeta, \zeta') \mapsto J(R_x)(\zeta, \zeta')$ is in $\cup_{\kappa, w_0, w_1, \varepsilon_0, \varepsilon_1} \mathcal{O}_{\sigma, \kappa, \varepsilon_0, \varepsilon_1, 0}^{0, w_0, w_1}(\mathbb{R})$ and $(x, \zeta, \zeta') \mapsto \tau(x, r_x(\zeta, \zeta'))$ is in $\mathcal{O}_{\sigma, \kappa_\tau, \kappa_\tau, \varepsilon_v/2, 0, z}^{0,0,0}$ for $\kappa_\tau \geq 0$.

Proof. (i) We have $s_{x,\zeta}(\zeta') = \sum_{i=1}^n \zeta'_i \int_0^1 \partial_{\zeta'_i} r_{x,\zeta}(t\zeta') dt$. Since $V \in \mathcal{O}_{\sigma,\kappa_v,\varepsilon_v,0}^{0,0,w_v}(\mathcal{M}_n(\mathbb{R}))$ each function $(x, \zeta, \zeta') \mapsto \int_0^1 \partial_{\zeta'_i} r_{x,\zeta}(t\zeta') dt$ is in $\mathcal{O}_{\sigma,\kappa_v,\varepsilon_v,0}^{0,0,w_v}(\mathbb{R}^n)$ and thus, since $(x, \zeta, \zeta') \mapsto \zeta'_i \in \mathcal{O}_{\sigma,\kappa_v,\varepsilon_v,1}^{0,0,1}(\mathbb{R})$, we see that $s \in \mathcal{O}_{\sigma,\kappa_v,\varepsilon_v,1}^{0,0,w_s}(\mathbb{R}^n)$. We have also $\varphi_{x,\zeta}(\zeta') = \sum_{|\beta|=2} \frac{2}{\beta!} (\zeta')^\beta \int_0^1 (1-t) \partial_{\zeta'}^\beta r_{x,\zeta}(t\zeta') dt$ and each function $(x, \zeta, \zeta') \mapsto \int_0^1 (1-t) \partial_{\zeta'}^\beta r_{x,\zeta}(t\zeta') dt$ is in $\mathcal{O}_{\sigma,\kappa_v,\varepsilon_v,0}^{-\varepsilon_v,\varepsilon_v,w_v+\kappa_v}(\mathbb{R}^n)$. With $(x, \zeta, \zeta') \mapsto (\zeta')^\beta \in \mathcal{O}_{\sigma,\kappa_v,\varepsilon_v,2}^{0,0,2}(\mathbb{R})$, we get $\varphi \in \mathcal{O}_{\sigma,\kappa_v,\varepsilon_v,2}^{-\varepsilon_v,\varepsilon_v,w_\varphi}(\mathbb{R}^n)$.

(ii) Direct consequence of (C_σ) and the following equalities for any $(x, \zeta, \zeta') \in \mathbb{R}^{3n}$, $(dr_{x,\zeta})_{\zeta'} = (d\bar{\psi}_x)_{x,\zeta,\zeta'}(d\psi_{\psi_x(\zeta)})_{\zeta'}$ and $(dr_{x,\zeta})_{\zeta'}^{-1} = (d\bar{\psi}_{\psi_x(\zeta)})_{x,\zeta,\zeta'}(d\psi_x)_{r_{x,\zeta}(\zeta')}$.

(iii) The first statement follows from Lemma 5.4.38 (ii). The second statement follows from Lemma 5.4.38 (i) and the estimate $r_x(\zeta, \zeta') = \mathcal{O}(\langle \zeta \rangle \langle \zeta' \rangle^{w_v})$. \square

We shall use a generalization to four variables of the $\Pi_{\sigma,\kappa,z}^{l,w,m}$ spaces of amplitude. We define $\tilde{\Pi}_{\sigma,\kappa,\varepsilon_1,z}^{l,w_0,w_1,m}$ ($0 < \varepsilon_1 \leq 1$) as the space of smooth functions $a \in C^\infty(\mathbb{R}^{4n}, L(E_z))$ such that for any $4n$ -multi-index $(\nu, \delta) \in \mathbb{N}^{3n} \times \mathbb{N}^n$, (with $\nu = (\mu, \gamma) \in \mathbb{N}^{2n} \times \mathbb{N}^n$) there is $C_{\nu,\delta} > 0$ such that for any $(x, \zeta, \zeta', \vartheta) \in \mathbb{R}^{4n}$,

$$\left\| \partial^{(\nu,\delta)} a(x, \zeta, \zeta', \vartheta) \right\|_{L(E_z)} \leq C_{\nu,\delta} \langle x \rangle^{\sigma(l-|\mu|-\varepsilon_1|\gamma|)} \langle \zeta \rangle^{w_0+\kappa|\nu|} \langle \zeta' \rangle^{w_1+\kappa|\nu|} \langle \vartheta \rangle^{m-|\delta|}.$$

These spaces have natural Fréchet topologies and form a graded topological algebra under point-wise composition.

Lemma 5.4.40. (i) If $a \in \tilde{\Pi}_{\sigma,\kappa,\varepsilon_1,z}^{l,w_0,w_1,m}$, then $a_{\zeta'=0} : (x, \zeta, \vartheta) \mapsto a(x, \zeta, 0, \vartheta)$ is in $\Pi_{\sigma,\kappa,z}^{l,w_0,m}$.

(ii) If $h \in \mathcal{O}_{\sigma,\kappa,\varepsilon_0,\varepsilon_1,0,z}^{l,w_0,w_1}$, then $(x, \zeta, \zeta', \vartheta) \mapsto h(x, \zeta, \zeta')$ is in $\tilde{\Pi}_{\sigma,\max\{\kappa,\varepsilon_0\},\varepsilon_1,z}^{l,w_0,w_1,0}$.

(iii) There is $\kappa_\Xi, k_1 \geq 0$ such that for any $b \in S_{\sigma,z}^{l,m}$, the application $b \circ \tilde{\Xi}$, where $\tilde{\Xi}(x, \zeta, \zeta', \vartheta) := (x^{\zeta,\zeta'}, -\tilde{P}_{-1,\psi(x,\zeta),\zeta'}^{z,b}(\vartheta))$, is in $\tilde{\Pi}_{\sigma,\kappa_\Xi,1,z}^{l,\sigma k_1|l|,\sigma k_1|l|,m}$.

Proof. (i) and (ii) are direct.

(iii) If $\mu = (\nu, \delta) \neq 0$ is a $4n$ -multi-index, we have $\partial^\mu(b \circ \tilde{\Xi}) = \sum_{1 \leq |\mu'| \leq |\mu|} P_{\mu,\mu'}(\tilde{\Xi})(\partial^{\mu'} b) \circ \tilde{\Xi}$ with $P_{\mu,\mu'}(\tilde{\Xi})$ a linear combination of terms of the form $\prod_{j=1}^s (\partial^{l^j} \tilde{\Xi})^{k^j}$, with $1 \leq s \leq |\mu|$, $l^j = (l^{j,1}, l^{j,2}) \in \mathbb{N}^{3n} \times \mathbb{N}^n$, $k^j = (k^{j,1}, k^{j,2}) \in \mathbb{N}^n \times \mathbb{N}^n$, such that $l^{j,2} = 0$ for $1 \leq j \leq j_1 \leq s$, and $\sum_1^s l^j |k^j| = \mu$, $\sum_1^s k^j = \mu'$. We have

$$(\partial^{l^j} \tilde{\Xi})^{k^j} = \prod_{i=1}^n (\delta_{l^{j,2},0} (\partial^{l^{j,1}} x^{\zeta,\zeta'})_i)^{k_i^{j,1}} \prod_{i=1}^n \left(\sum_{k=1}^n \partial^{l^{j,1}} P^{i,k} \partial^{l^{j,2}} \vartheta_k \right)^{k_i^{j,2}}$$

where $P^{i,k}$ are the matrix entries of $-\tilde{P}_{-1,\psi(x,\zeta),\zeta'}^{z,b}$. By Lemma 5.4.38 (ii) and (iii), $x^{\zeta,\zeta'} \in \mathcal{R}_{\sigma,\kappa_\psi,1}^{w_0,w_1}(\mathbb{R}^n)$ and the $P^{i,k}$ are in $\mathcal{O}_{\sigma,\kappa_\psi,\kappa_\psi,1,0}^{0,0,0}(\mathbb{R})$ for a $(\kappa_\psi, w_0, w_1) \in \mathbb{R}_+^3$. We obtain thus the following estimate

$$|P_{\mu,\mu'}(\tilde{\Xi})(x, \zeta, \zeta', \vartheta)| \leq C_\mu \langle x \rangle^{-\sigma(|\nu|-|\alpha'|)} \langle \zeta \rangle^{w_0|\alpha'|+\kappa_\psi(|\nu|-|\alpha'|)} \langle \zeta' \rangle^{w_1|\alpha'|+\kappa_\psi(|\nu|-|\alpha'|)} \langle \vartheta \rangle^{|\beta'|-|\delta|}$$

with $\mu' =: (\alpha', \beta')$. Since $b \in S_{\sigma,z}^{l,m}$ we also have the estimate

$$\left\| (\partial^{\mu'} b) \circ \tilde{\Xi}(x, \zeta, \zeta') \right\| \leq C'_\mu \langle x^{\zeta,\zeta'} \rangle^{\sigma(l-|\alpha'|)} \langle \vartheta \rangle^{m-|\beta'|}$$

so the result follows now from the estimate $\langle x^{\zeta,\zeta'} \rangle^{\sigma(l-|\alpha'|)} = \mathcal{O}(\langle x \rangle^{\sigma(l-|\alpha'|)} (\langle \zeta \rangle \langle \zeta' \rangle)^{\sigma k_1|l|+\sigma k_1|\alpha'|})$, with $\kappa_\Xi := \kappa_\psi + \max\{|w_0 + \sigma k_1 - \kappa_\psi|, |w_1 + \sigma k_1 - \kappa_\psi|\}$. \square

Lemma 5.4.41. *Let $s \in C^\infty(\mathbb{R}^p, \mathbb{R}^n)$. Then for any $p + n$ -multi-index $\nu = (\alpha, \beta) \neq 0$, we have*

$$\partial_{x,\vartheta}^\nu e^{i\langle \vartheta, s(x) \rangle} = P_\nu(x, \vartheta) e^{i\langle \vartheta, s(x) \rangle}$$

where P_ν is of the form $\sum_{|\gamma| \leq |\alpha|} \vartheta^\gamma T_{\nu, \gamma}(x)$, and $T_{\nu, \gamma}$ is a linear combination of terms of the form $\prod_{j=1}^m (\partial^{l^j} s)^{\mu^j}$ where $1 \leq m \leq |\nu|$, (l^j) are p -multi-indices and (μ^j) are n -multi-indices. Moreover, they satisfy $|\mu^j| > 0$, $\sum_{j=1}^m |\mu^j| = |\gamma| + |\beta|$, $\sum_{j=1}^m |\mu^j| |l^j| = |\alpha|$ and if $|\beta| = 0$, then $|l^j| > 0$ and $|\gamma| > 0$.

Proof. We note $g(x, \vartheta) := \langle \vartheta, s(x) \rangle$. By Theorem 5.2.11, we get the following equality for any $\nu \neq 0$, $\partial_{x,\vartheta}^\nu e^{i\langle \vartheta, s(x) \rangle} = P_\nu(x, \vartheta) e^{i\langle \vartheta, s(x) \rangle}$ where $P_\nu(x, \vartheta) = \sum_{1 \leq k \leq |\nu|} P_{\nu, k}(g)$ and $P_{\nu, k}$ is a linear combination of terms of the form $\prod_{j=1}^m (\partial^{l^j} g)^{k^j}$ such that $|l^j| > 0$, $k^j > 0$, $\sum_1^m k^j = k$ and $\sum_1^m k^j l^j = \nu$. If we suppose that the term $\prod_{j=1}^m (\partial^{l^j} g)^{k^j}$ is non-zero, then $|l^j| \leq 1$ and if we define j_1 such that for any $1 \leq j \leq j_1$, $l^{j,2} = 0$, we obtain, noting $l^j = (l^{j,1}, l^{j,2})$,

$$\begin{aligned} \prod_{j=1}^m (\partial^{l^j} g)^{k^j} &= \prod_{j=1}^{j_1} \langle \vartheta, \partial^{l^{j,1}} s \rangle^{k^j} \prod_{j=j_1+1}^m (\partial^{l^{j,1}} s^{q_j})^{k^j} \\ &= \sum_{|\gamma^j| = k^j, 1 \leq j \leq j_1} \gamma^1! \dots \gamma^{j_1}! \vartheta^{\sum_{j=1}^{j_1} \gamma^j} \prod_{j=1}^{j_1} (\partial^{l^{j,1}} s)^{\gamma^j} \prod_{j=j_1+1}^m (\partial^{l^{j,1}} s^{q_j})^{k^j}. \end{aligned}$$

Thus, we have $P_{\nu, k} = \sum_{|\gamma| = k - |\beta|} \vartheta^\gamma T_{\nu, \gamma, k}(x)$ where $T_{\nu, \gamma, k}$ is a linear combination of terms of the form $\prod_{j=1}^{j_1} (\partial^{l^{j,1}} s)^{\mu^j} \prod_{j=j_1+1}^m (\partial^{l^{j,1}} s^{q_j})^{k^j}$, where $1 \leq q_j \leq n$, $1 \leq j \leq m \leq |\nu|$, $1 \leq j_1 \leq m$, $l^{j,1} \in \mathbb{N}^p$, $k^j \in \mathbb{N}^*$, $\lambda^j \in \mathbb{N}^n$ are such that $\sum_1^m k^j = k$, $\sum_1^{j_1} |\lambda^j| |l^{j,1}| + \sum_{j_1+1}^m k^j |l^{j,1} + 1| = |\nu|$ and $\sum_{j_1+1}^m k^j = |\beta|$. The result follows. \square

Lemma 5.4.42. *Suppose that (C_σ) is satisfied. Then*

(i) *Representing by u the letter s or φ , for any $3n$ -multi-index $\nu = (\mu, \gamma) \in \mathbb{N}^{2n} \times \mathbb{N}^n$, we have the equality $\partial_{x,\zeta,\vartheta}^\nu e^{2\pi i \langle \vartheta, u_{x,\zeta}(\zeta') \rangle} = (\sum_{|\omega| \leq |\mu|} \vartheta^\omega T_{\nu, \omega, u}(x, \zeta, \zeta')) e^{2\pi i \langle \vartheta, u_{x,\zeta}(\zeta') \rangle}$ where each term $T_{\nu, \omega, s} \in \mathcal{O}_{\sigma, \kappa_v, \varepsilon_v, \varepsilon_v, |\omega + \gamma|}^{-|\mu|, \kappa_v |\mu|, w_s |\omega + \gamma| + \kappa_v |\mu|}(\mathbb{R})$ and $T_{\nu, \omega, \varphi} \in \mathcal{O}_{\sigma, \kappa_v, \varepsilon_v, \varepsilon_v, 2|\omega + \gamma|}^{-|\mu| - \varepsilon_v |\omega + \gamma|, \varepsilon_v |\omega + \gamma| + \kappa_v |\mu|, w_\varphi |\omega + \gamma| + \kappa_v |\mu|}(\mathbb{R})$. In particular, it satisfies the following estimate valid for any $(x, \zeta, \zeta') \in \mathbb{R}^{3n}$, and any n -multi-index ρ ,*

$$\begin{aligned} |\partial_{\zeta'}^\rho T_{\nu, \omega, s}(x, \zeta, \zeta')| &\leq C_{\nu, \omega, \rho} \langle x \rangle^{-\sigma(|\mu| + \varepsilon_v |\rho| |\omega + \gamma|)} \langle \zeta \rangle^{\kappa_v |\mu| + \varepsilon_v |\rho|} \langle \zeta' \rangle^{w_s |\omega + \gamma| + \kappa_v (|\mu| + |\rho|)}, \\ |\partial_{\zeta'}^\rho T_{\nu, \omega, \varphi}(x, \zeta, \zeta')| &\leq C_{\nu, \omega, \rho} \langle x \rangle^{-\sigma(|\mu| + (\varepsilon_v/2) |\rho|)} \langle \zeta \rangle^{\varepsilon_v |\omega + \gamma| + \kappa_v |\mu| + \varepsilon_v |\rho|} \langle \zeta' \rangle^{w_\varphi |\omega + \gamma| + \kappa_v (|\mu| + |\rho|)}. \end{aligned}$$

(ii) *For any n -multi-index β , we have $\partial_\zeta^\beta e^{2\pi i \langle \vartheta, \varphi_{x,\zeta}(\zeta') \rangle} = P_{\beta, \varphi}(x, \zeta, \zeta', \vartheta) e^{2\pi i \langle \vartheta, \varphi_{x,\zeta}(\zeta') \rangle}$ where $P_{\beta, \varphi}(x, \zeta, \zeta', \vartheta)$ is a linear combination of terms of the form $\vartheta^\omega \zeta'^\lambda t_{\omega, \lambda}(x, \zeta, \zeta')$ where ω and λ are n -multi-indices satisfying $|\omega| \leq |\beta|$, $(2|\omega| - |\beta|)_+ \leq |\lambda| \leq |\omega|$, and $t_{\omega, \lambda}$ are functions in $\mathcal{O}_{\sigma, \kappa_v, \varepsilon_v, \varepsilon_v, |\beta|}^{-\varepsilon_v |\beta|/2, 2\varepsilon_v, w'_s |\beta|}(\mathbb{R})$. In particular they are estimated by*

$$t_{\omega, \lambda}(x, \zeta, \zeta') = \mathcal{O}(\langle x \rangle^{-\sigma \varepsilon_v |\beta|/2} \langle \zeta \rangle^{2\varepsilon_v |\beta|} \langle \zeta' \rangle^{w'_s |\beta|})$$

where $w'_s := w_s + 2\kappa_v$. Moreover, $(x, \zeta, \vartheta) \mapsto P_{\beta, \varphi}(x, \zeta, 0, \vartheta) 1_{L(E_z)} \in \Pi_{\sigma, \kappa_v, z}^{-\varepsilon_v |\beta|/2, \varepsilon_v |\beta|, |\beta|/2}$.

(iii) If $\beta \in \mathbb{N}^n$ and $f \in \tilde{\Pi}_{\sigma, \kappa, \varepsilon_1, z}^{l, w_0, w_1, m}$ then the function

$$f_{\beta, \varphi} : (x, \zeta, \vartheta) \mapsto \partial_{\zeta'}^{\beta} (e^{2\pi i \langle \vartheta, \varphi_{x, \zeta}(\zeta') \rangle} \partial^{0, 0, 0, \beta} f(x, \zeta, \zeta', L_{x, \zeta}(\vartheta)))_{\zeta'=0}$$

belongs to $\Pi_{\sigma, \kappa_1, z}^{l - \varepsilon'_1 |\beta|, w_0 + \kappa_2 |\beta|, m - |\beta|/2}$, where $\varepsilon'_1 := \min\{\varepsilon_1/2, \varepsilon_v/2\} > 0$, $\kappa_1 := \max\{\kappa_v, \kappa\}$, $\kappa_2 := \kappa + |\varepsilon_v - \kappa|$, and the application $f \mapsto f_{\beta, \varphi}$ is continuous.

Proof. (i) By Lemma 5.4.41, if $\nu \neq 0$, we have the following equality, valid for any $(x, \zeta, \zeta', \vartheta) \in \mathbb{R}^{4n}$, $\partial_{x, \zeta, \vartheta}^{\nu} e^{2\pi i \langle \vartheta, u_{x, \zeta}(\zeta') \rangle} = (\sum_{|\omega| \leq |\mu|} \vartheta^{\omega} T_{\nu, \omega, u}(x, \zeta, \zeta')) e^{2\pi i \langle \vartheta, u_{x, \zeta}(\zeta') \rangle}$ where $T_{\nu, \omega, u}$ is a linear combination of terms of the form $\prod_{j=1}^m (\partial_{x, \zeta}^{l^j} u)^{\mu^j}$ with $1 \leq m \leq |\nu|$, $\mu^j \neq 0$, $\sum_{j=1}^m |\mu^j| = |\omega + \gamma|$ and $\sum_{j=1}^m |\mu^j| |l^j| = |\mu|$. Since by Lemma 5.4.39 (i), $s \in \mathcal{O}_{\sigma, \kappa_v, \varepsilon_v, \varepsilon_v, 1}^{0, 0, w_s}(\mathbb{R}^n)$, it is straightforward to check that $T_{\nu, \omega, s} \in \mathcal{O}_{\sigma, \kappa_v, \varepsilon_v, \varepsilon_v, |\omega + \gamma|}^{-|\mu|, \kappa_v |\mu|, w_s |\omega + \gamma| + \kappa_v |\mu|}(\mathbb{R})$. Moreover, since $\varphi \in \mathcal{O}_{\sigma, \kappa_v, \varepsilon_v, \varepsilon_v, 2}^{-\varepsilon_v, \varepsilon_v, w_{\varphi}}(\mathbb{R}^n)$, we get $T_{\nu, \omega, \varphi} \in \mathcal{O}_{\sigma, \kappa_v, \varepsilon_v, \varepsilon_v, 2|\omega + \gamma|}^{-|\mu| - \varepsilon_v |\omega + \gamma|, \varepsilon_v |\omega + \gamma| + \kappa_v |\mu|, w_{\varphi} |\omega + \gamma| + \kappa_v |\mu|}(\mathbb{R})$. The first estimate is direct and the second estimate follows from the inequality $|\omega + \gamma| + |\rho|_{2|\omega + \gamma|} \geq |\rho|/2$.

(ii) By Lemma 5.4.41, if $\beta \neq 0$, we have for any $(x, \zeta, \zeta', \vartheta) \in \mathbb{R}^{4n}$, the following relation $\partial_{\zeta'}^{\beta} e^{2\pi i \langle \vartheta, \varphi_{x, \zeta}(\zeta') \rangle} = (\sum_{1 \leq |\omega| \leq |\beta|} \vartheta^{\omega} T_{\beta, \omega, \varphi}(x, \zeta, \zeta')) e^{2\pi i \langle \vartheta, \varphi_{x, \zeta}(\zeta') \rangle}$ where $T_{\beta, \omega, \varphi}$ is a linear combination of terms of the form $\prod_{j=1}^m (\partial^{l^j} \varphi_{x, \zeta})^{\mu^j}$ with $1 \leq m \leq |\beta|$, $\mu^j \neq 0$, $l^j \neq 0$, $\sum_{j=1}^m |\mu^j| = |\omega|$ and $\sum_{j=1}^m |\mu^j| |l^j| = |\beta|$. Let us reorder the l^j indices so that for any $1 \leq j \leq j_1$, $|l^j| = 1$ and for any $j \geq j_1 + 1$, $|l^j| > 1$, where $j_1 \in \{0, \dots, m\}$. Thus $\prod_{j=1}^m (\partial^{l^j} \varphi_{x, \zeta})^{\mu^j} = \prod_{j=1}^{j_1} (\partial^{l^j} \varphi_{x, \zeta})^{\mu^j} \prod_{j \geq j_1 + 1} (\partial^{l^j} \varphi_{x, \zeta})^{\mu^j}$ and with a Taylor expansion at order 1 of $\partial^{l^j} \varphi_{x, \zeta}$ in ζ' around 0 when $1 \leq j \leq j_1$, we get $\partial^{l^j} \varphi_{x, \zeta} = \sum_{1 \leq i \leq n} \zeta'_i t_{i, j}^k$ where $t_{i, j}^k = \int_0^1 \partial_{\zeta'}^{e_i + l^j} \varphi_{x, \zeta}(t \zeta') dt$. Thus, using the fact that $\varphi \in \mathcal{O}_{\sigma, \kappa_v, \varepsilon_v, \varepsilon_v, 1}^{0, 0, w_s}(\mathbb{R}^n)$, we see that $\prod_{j=1}^{j_1} (\partial^{l^j} \varphi_{x, \zeta})^{\mu^j}$ is a linear combination of terms of the form $\zeta'^{\lambda} V_{\lambda}$ where $|\lambda| = \sum_{j=1}^{j_1} |\mu^j|$ and

$$V_{\lambda} = \mathcal{O}(\langle x \rangle^{-\sigma \varepsilon_v} \sum_{j=1}^{j_1} |\mu^j| \langle \zeta \rangle^{\varepsilon_v |\lambda| + \varepsilon_v} \sum_{j=1}^{j_1} |\mu^j| |l^j| \langle \zeta' \rangle^{(\kappa_v + w_s) |\lambda| + \kappa_v} \sum_{j=1}^{j_1} |\mu^j| |l^j|).$$

As a consequence, we see that $\prod_{j=1}^m (\partial^{l^j} \varphi_{x, \zeta})^{\mu^j}$ is a linear combination of terms of the form $\zeta'^{\lambda} W_{\lambda}$ where $|\lambda| = \sum_{j=1}^{j_1} |\mu^j|$ and

$$W_{\lambda} = \mathcal{O}(\langle x \rangle^{-\sigma \varepsilon_v (|\beta| - v)} \langle \zeta \rangle^{2\varepsilon_v |\beta|} \langle \zeta' \rangle^{w'_s |\beta|})$$

where $v := \sum_{j=j_1+1}^m |\mu^j| = |\omega| - |\lambda|$. The first statement now follows from the inequality $2v \leq |\beta| - |\lambda|$.

Since $\varphi_{x, \zeta}(0) = 0$ and $(d\varphi_{x, \zeta})_0 = 0$, $P_{\beta, \varphi}(x, \zeta, 0, \vartheta)$ is a linear combination of terms of the form $\vartheta^{\omega} \prod_{j=1}^m (\partial^{0, 0, l^j} \varphi(x, \zeta, 0))^{\mu^j}$ with $1 \leq |\omega| \leq |\beta|/2$, $1 \leq m \leq |\beta|$, $\mu^j \neq 0$, $|l^j| \geq 2$, $\sum_{j=1}^m |\mu^j| = |\omega|$ and $\sum_{j=1}^m |\mu^j| |l^j| = |\beta|$. We check with Lemma 5.4.39 (i) that any function of the form $\prod_{j=1}^m (\partial^{0, 0, l^j} \varphi(x, \zeta, \zeta'))^{\mu^j}$ is in $\mathcal{O}_{\sigma, \kappa_v, \varepsilon_v, |\beta|/2}^{-\varepsilon_v |\beta|/2, \varepsilon_v |\beta|, (w_s/2 + \kappa_v) |\beta|}(\mathbb{R})$, and thus, $(x, \zeta, \vartheta) \mapsto \prod_{j=1}^m (\partial^{0, 0, l^j} \varphi(x, \zeta, 0))^{\mu^j} 1_{L(E_z)} \in \Pi_{\sigma, \kappa_v, z}^{-\varepsilon_v |\beta|/2, \varepsilon_v |\beta|, 0}$. Since $(x, \zeta, \vartheta) \mapsto \vartheta^{\omega} 1_{L(E_z)} \in \Pi_{\sigma, \kappa_v, z}^{0, 0, |\beta|/2}$ we obtain $(x, \zeta, \vartheta) \mapsto P_{\beta, \varphi}(x, \zeta, 0, \vartheta) 1_{L(E_z)} \in \Pi_{\sigma, \kappa_v, z}^{-\varepsilon |\beta|/2, \varepsilon_v |\beta|, |\beta|/2}$.

(iii) We have

$$\begin{aligned} f_{\beta, \varphi}(x, \zeta, \vartheta) &= \sum_{\beta' \leq \beta} \binom{\beta}{\beta'} \partial_{\zeta'}^{\beta'} (e^{2\pi i \langle \vartheta, \varphi_{x, \zeta}(\zeta') \rangle})_{\zeta'=0} \partial^{0, 0, \beta - \beta', \beta} f(x, \zeta, 0, L_{x, \zeta}(\vartheta)) \\ &= \sum_{\beta' \leq \beta} \binom{\beta}{\beta'} P_{\beta', \varphi}(x, \zeta, 0, \vartheta) \partial^{0, 0, \beta - \beta', \beta} f(x, \zeta, 0, L_{x, \zeta}(\vartheta)). \end{aligned}$$

Since $(x, \zeta) \mapsto L_{x, \zeta} \in E_{\sigma, \kappa_v}^0(\mathcal{M}_n(\mathbb{R}))$ and $L_{x, \zeta}^{-1} = \mathcal{O}(1)$, we deduce from Lemma 5.4.40 (i) and Lemma 5.4.23 (iv) that $(x, \zeta, \vartheta) \mapsto \partial^{0,0,\beta-\beta',\beta} f(x, \zeta, 0, L_{x, \zeta}(\vartheta))$ belongs to the amplitude space $\Pi_{\sigma, \max\{\kappa, \kappa_v\}, z}^{l-\varepsilon_1|\beta-\beta'|, w_0+\kappa|\beta-\beta'|, m-|\beta|}$. The result now follows from (ii). \square

We now introduce two parametrized cut-off functions that will be used later. Let $b \in C_c^\infty(\mathbb{R}, [0, 1])$ such that $b = 1$ on $[-1/4, 1/4]$ and $b = 0$ on $\mathbb{R} \setminus]-1, 1[$. We define for $\varepsilon, \delta, \eta_1, \eta_2 > 0$ with $\varepsilon, \delta < 1$,

$$\chi_\varepsilon(\vartheta, \vartheta') := b\left(\frac{\|\vartheta'\|^2}{\varepsilon^2 \langle \vartheta \rangle^2}\right),$$

$$\chi_{\delta, \eta}(x, \zeta, \zeta') := b\left(\frac{\|\zeta'\|^2}{\delta^2 \langle x \rangle^{2\sigma\eta_1} \langle \zeta \rangle^{-2\eta_2}}\right).$$

Lemma 5.4.43. *The cut-off functions χ_ε and $\chi_{\delta, \eta}$ are respectively in the spaces $C^\infty(\mathbb{R}^{2n}, [0, 1])$ and $C^\infty(\mathbb{R}^{3n}, [0, 1])$ and satisfy:*

- (i) *For any $(x, \zeta, \zeta') \in \mathbb{R}^{3n}$, if $\|\zeta'\| \leq \frac{1}{2}\delta \langle x \rangle^{\sigma\eta_1} \langle \zeta \rangle^{-\eta_2}$, then $\chi_{\delta, \eta}(x, \zeta, \zeta') = 1$, and if $\|\zeta'\| \geq \delta \langle x \rangle^{\sigma\eta_1} \langle \zeta \rangle^{-\eta_2}$, then $\chi_{\delta, \eta}(x, \zeta, \zeta') = 0$. In particular, for any $(x, \zeta) \in \mathbb{R}^{2n}$, $\chi_{\delta, \eta}(x, \zeta, 0) = 1$ and for any $3n$ -multi-index $\nu \neq 0$, $(\partial^\nu \chi_{\delta, \eta})(x, \zeta, 0) = 0$.*
- (ii) *For any $(\vartheta, \vartheta') \in \mathbb{R}^{2n}$, if $\|\vartheta'\| \leq \frac{1}{2}\varepsilon \langle \vartheta \rangle$, then $\chi_\varepsilon(\vartheta, \vartheta') = 1$, and if $\|\vartheta'\| \geq \varepsilon \langle \vartheta \rangle$, then $\chi_\varepsilon(\vartheta, \vartheta') = 0$. In particular, for any $\vartheta \in \mathbb{R}^n$, $\chi_\varepsilon(\vartheta, 0) = 1$ and for any $2n$ -multi-index $\nu \neq 0$, $(\partial^\nu \chi_\varepsilon)(\vartheta, 0) = 0$.*
- (iii) *For any $3n$ -multi-index $\nu = (\alpha, \beta, \gamma)$, we have $\partial^\nu \chi_{\delta, \eta}(x, \zeta, \zeta') = \mathcal{O}(\langle x \rangle^{-|\alpha|} \langle \zeta \rangle^{-\beta} \langle \zeta' \rangle^{-|\gamma|})$, and $\partial^\nu \chi_{\delta, \eta}(x, \zeta, \zeta') = \mathcal{O}(\langle x \rangle^{-\sigma|\nu|} \langle \zeta \rangle^{(-1+\eta_2/\eta_1)|\beta|+(\eta_2/\eta_1)|\gamma|} \langle \zeta' \rangle^{(\eta_1^{-1}-1)|\gamma|+\eta_1^{-1}|\beta|})$. In particular, the function $\chi_{\delta, \eta}$ is in $\mathcal{O}_{\sigma, \kappa'_\eta, \kappa'_\eta, 1, 0}^{0,0,0}(\mathbb{R})$ for a $\kappa'_\eta > 0$.*
- (iv) *For any $2n$ -multi-index ν , $\partial^\nu \chi_\varepsilon(\vartheta, \vartheta') = \mathcal{O}(\langle \vartheta \rangle^{-|\nu|})$ and $\partial^\nu \chi_\varepsilon(\vartheta, \vartheta') = \mathcal{O}(\langle \vartheta' \rangle^{-|\nu|})$.*

Proof. (i) and (ii) are straightforward. For any $\nu \neq 0$, $\partial^\nu \chi_{\delta, \eta} = \sum_{1 \leq \nu' \leq |\nu|} P_{\nu, \nu'}(g) (\partial^{\nu'} b) \circ g$ where $g(x, \zeta, \zeta') := \frac{\|\zeta'\|^2}{\delta^2 \langle x \rangle^{2\sigma\eta_1} \langle \zeta \rangle^{-2\eta_2}}$. We obtain from a direct computation the estimate $P_{\nu, \nu'}(g) = \mathcal{O}(\langle x \rangle^{-2\sigma\eta_1 \nu' - |\alpha|} \langle \zeta \rangle^{2\eta_2 \nu' - |\beta|} \langle \zeta' \rangle^{2\nu' - |\gamma|})$. Since for any $\nu \in \mathbb{N}$, we have $\partial^{\nu'} b = \mathcal{O}(1)$ we obtain $\partial^\nu \chi_{\delta, \eta} = \mathcal{O}(\langle x \rangle^{-|\alpha|} \langle \zeta \rangle^{-\beta} \langle \zeta' \rangle^{-|\gamma|} 1_{D_\delta})$ where D_δ is the set of triples (x, ζ, ζ') satisfying the inequalities $\delta/2 \leq \langle \zeta' \rangle \langle x \rangle^{-\sigma\eta_1} \langle \zeta \rangle^{\eta_2} \leq \sqrt{2}$. The estimates of (iii) follow. The proof of (iv) is similar. \square

We will use in the following lemma the space $\mathcal{O}_\kappa^{t_0, t_1, j}$ ($\kappa \geq 0, j \in \mathbb{N}, (t_0, t_1) \in \mathbb{R}_+^2$) of functions $f \in C^\infty(\mathbb{R}^{4n}, \mathbb{C})$ such that for any $\alpha \in \mathbb{N}^n$, there is $C_\alpha > 0$ such that for any $(x, \zeta, \zeta', \vartheta) \in \mathbb{R}^{4n}$, $|\partial_{\zeta'}^\alpha f(x, \zeta, \zeta', \vartheta)| \leq C_\alpha \langle \zeta \rangle^{t_0+\kappa|\alpha|} \langle \zeta' \rangle^{t_1+\kappa|\alpha|} \langle \vartheta \rangle^{-2j}$. Clearly, $\mathcal{O}_\kappa^{t_0, t_1, j} \mathcal{O}_\kappa^{t'_0, t'_1, j'} \subseteq \mathcal{O}_\kappa^{t_0+t'_0, t_1+t'_1, j+j'}$ and $\partial_{\zeta'}^\alpha \mathcal{O}_\kappa^{t_0, t_1, j} \subseteq \mathcal{O}_\kappa^{t_0+\kappa|\alpha|, t_1+\kappa|\alpha|, j}$.

Lemma 5.4.44. *Defining $h(x, \zeta, \zeta', \vartheta) := (1 + \|{}^t(ds_{x, \zeta})_{\zeta'}(\vartheta)\|^2 - (i/2\pi)\langle \vartheta, (\Delta s_{x, \zeta})(\zeta') \rangle)^{-1}$, we have the following relation, valid for any $(x, \zeta, \zeta', \vartheta) \in \mathbb{R}^{4n}$, $p \in \mathbb{N}$,*

$$e^{2\pi i \langle \vartheta, s_{x, \zeta}(\zeta') \rangle} = (h(x, \zeta, \zeta', \vartheta) L_{\zeta'})^p e^{2\pi i \langle \vartheta, s_{x, \zeta}(\zeta') \rangle}$$

where $L_{\zeta'} := 1 - (2\pi)^{-2} \Delta_{\zeta'}$. Moreover, if (C_σ) holds, there is $\kappa_L \geq 0$ such that for any $p \in \mathbb{N}$, there is $N_p \in \mathbb{N}^*$, $(h_k^p)_{1 \leq k \leq N_p}$ functions in $\mathcal{O}_{\kappa_L}^{2p\kappa_L, 2p\kappa_L, p}$, $(\beta^{k,p})_{1 \leq k \leq N_p}$ n -multi-indices satisfying $|\beta^{k,p}| \leq 2p$, such that $(L_{\zeta'} h)^p = \sum_{k=1}^{N_p} h_k^p \partial_{\zeta'}^{\beta^{k,p}}$.

Proof. We obtain $L_{\zeta'} e^{2\pi i \langle \vartheta, s_{x,\zeta}(\zeta') \rangle} = (1/h) e^{2\pi i \langle \vartheta, s_{x,\zeta}(\zeta') \rangle}$ through a direct computation. Let us show the remaining statement by induction on p . Note that by Lemma 5.4.39 (ii), we have $|1/h| \geq c \langle \vartheta \rangle^2$ for a $c > 0$ and we check that $1/h \in \tilde{\Pi}_{\sigma, \kappa_v, \varepsilon_v, z}^{0, \varepsilon_v, w'_v, 2}$ where $w'_v = \max\{2w_v, w_v + \kappa_v\}$. With a recurrence or using Proposition 5.5.4, we check that $h \in \mathcal{O}_{\kappa_L}^{0, 0, 1}$ where $\kappa_L := \max\{2\varepsilon_v, w'_v + \kappa_v\}$. The property is obviously true for $p = 0$. Suppose now that the property is true for $p \geq 0$, so that $(L_{\zeta'} h)^p = \sum_{k=1}^{N_p} h_k^p \partial_{\zeta'}^{\beta^{k,p}}$ with $N_p \in \mathbb{N}^*$, $(h_k^p)_{1 \leq k \leq N_p}$ functions in $\mathcal{O}_{\kappa_L}^{2p\kappa_L, 2p\kappa_L, p}$ and $(\beta^{k,p})_{1 \leq k \leq N_p}$ n -multi-indices satisfying $|\beta^{k,p}| \leq 2p$. We also have

$$\begin{aligned} (L_{\zeta'} h)^{p+1} &= (L_{\zeta'} h) \sum_{k=1}^{N_p} h_k^p \partial_{\zeta'}^{\beta^{k,p}} = \sum_{k=1}^{N_p} h h_k^p \partial_{\zeta'}^{\beta^{k,p}} - (2\pi)^{-2} (\Delta_{\zeta'} (h h_k^p) \partial_{\zeta'}^{\beta^{k,p}} \\ &\quad + 2 \sum_{i=1}^n \partial_{\zeta'_i} (h h_k^p) \partial_{\zeta'}^{\beta^{k,p} + e_i} + h h_k^p \Delta_{\zeta'} \partial_{\zeta'}^{\beta^{k,p}}) \end{aligned}$$

so the property holds for $p + 1$. \square

We note $\mathcal{S}_{\sigma, c}(\mathbb{R}^{3n}, L(E_z))$ the space of smooth functions f such that for any $N \in \mathbb{N}^*$ and $\nu = (\mu, \gamma) \in \mathbb{N}^{2n} \times \mathbb{N}^n$, $\partial^\nu f(x, \zeta, \vartheta) = \mathcal{O}(\langle x \rangle^{-\sigma N} \langle \zeta \rangle^{c_0 + c_1 N + c_2 |\mu|} \langle \vartheta \rangle^{-N})$. It follows from Lemma 5.4.18 that if $f \in \mathcal{S}_{\sigma, c}(\mathbb{R}^{3n}, L(E_z))$, then $\mathfrak{D}_{\mathbf{p}_\Gamma}(f) \in \mathfrak{D}_{\mathbf{p}_\Gamma}(\mathcal{S}_{\sigma, z}^{-\infty})$. Here and thereafter Γ satisfies the hypothesis of Lemma 5.4.18.

Lemma 5.4.45. *Assume that (C_σ) holds.*

(i) *For any l, w_0, w_1, m, κ , $S_{m, w_1}(\tilde{\Pi}_{\sigma, \kappa, \varepsilon_1, z}^{l, w_0, w_1, m}) \subseteq \mathcal{S}_{\sigma, c}(\mathbb{R}^{3n}, L(E_z))$ for a triple $c := (c_0, c_1, c_2)$ and the linear map $S_{m, w_1} : f \mapsto S_{m, w_1}(f)$ is continuous, where*

$$S_{m, w_1}(f) : (x, \zeta, \vartheta) \mapsto \int_{\mathbb{R}^{2n}} e^{2\pi i (\langle \vartheta', \zeta' \rangle + \langle \vartheta, s_{x, \zeta}(\zeta') \rangle)} {}^t M_{\vartheta'}^{p_{m, w_1}, \zeta'}(f)(x, \zeta, \zeta', \vartheta') (1 - \chi_{\delta, \eta})(x, \zeta, \zeta') d\vartheta' d\zeta'$$

and $p_{m, w_1} := \max\{m + 2n, \lfloor w_1 \rfloor + 1 + 2n\}$.

(ii) *For any $u \in \mathcal{S}(\mathbb{R}^{2n}, L(E_z))$, the linear application $f \mapsto \langle \mathfrak{D}_{\mathbf{p}_\Gamma} S_{m, w_1}(f), u \rangle$ is continuous.*

Proof. We fix $N \in \mathbb{N}^*$. First note that $S_{m, w_1}(f)$ is well-defined since for any $(x, \zeta) \in \mathbb{R}^{2n}$, there is $C_{x, \zeta} > 0$ such that $\| {}^t M_{\vartheta'}^{p_{m, w_1}, \zeta'}(f)(x, \zeta, \zeta', \vartheta') (1 - \chi_{\delta, \eta})(x, \zeta, \zeta') \| \leq C_{x, \zeta} \langle \vartheta' \rangle^{-2n} \langle \zeta' \rangle^{-2n}$. Since for any n -multi-index δ , $\partial_{\vartheta'}^\delta {}^t M_{\vartheta'}^{p_{m, w_1}, \zeta'}(f)$ decrease to zero with ϑ' , we can successively integrate by parts with (5.26), which is valid since $1 - \chi_{\delta, \eta}$ assures that $\|\zeta'\| \geq \frac{1}{2}\delta$ on the domain of integration. We obtain thus for any $q \in \mathbb{N}^*$,

$$S_{m, w_1}(f) : (x, \zeta, \vartheta) \mapsto \int_{\mathbb{R}^{2n}} e^{2\pi i (\langle \vartheta', \zeta' \rangle + \langle \vartheta, s_{x, \zeta}(\zeta') \rangle)} {}^t M_{\vartheta'}^{p_{m, w_1} + q, \zeta'}(f)(1 - \chi_{\delta, \eta}) d\vartheta' d\zeta'.$$

We note f_q the integrand of the previous integral. If $\nu = (\alpha, \beta, \gamma) = (\mu, \gamma)$ is a $3n$ -multi-index, we see with Lemma 5.4.41 that

$$\begin{aligned} \partial_{x, \zeta, \vartheta}^\nu f_q &= e^{2\pi i \langle \vartheta', \zeta' \rangle} \sum_{\mu' \leq \mu} \binom{\mu}{\mu'} e^{2\pi i \langle \vartheta, s_{x, \zeta}(\zeta') \rangle} \sum_{|\omega| \leq |\mu'|} \vartheta^\omega T_{\nu', \omega, s}(x, \zeta, \zeta') \\ &\quad \sum_{|\tilde{\delta}| = p_{m, w_1} + q} \lambda_\delta (-1)^{|\tilde{\delta}|} \frac{\zeta'^{\tilde{\delta}}}{\|\zeta'\|^{2(p_{m, w_1} + q)}} \partial_{x, \zeta}^{\mu - \mu'} \partial_{\vartheta'}^{\tilde{\delta}} (f(1 - \chi_{\delta, \eta})). \end{aligned}$$

By Lemma 5.4.43 (iii), $(x, \zeta, \zeta', \vartheta') \mapsto \chi_{\delta, \eta}(x, \zeta, \zeta') 1_{L(E_z)}$ is in $\tilde{\Pi}_{\sigma, \kappa'_\eta, 1, z}^{0, 0, 0, 0}$, so the multiplication operator $f \mapsto f(1 - \chi_{\delta, \eta})$ is continuous from $\tilde{\Pi}_{\sigma, \kappa, \varepsilon_1, z}^{l, w_0, w_1, m}$ into $\tilde{\Pi}_{\sigma, \kappa_\eta, \varepsilon_1, z}^{l, w_0, w_1, m}$, where $\kappa_\eta = \max\{\kappa, \kappa'_\eta\}$. Since $\|\zeta'\| \geq \delta/2$ in the support of $f(1 - \chi_{\delta, \eta})$, we get from Lemma 5.4.42 (i) the following estimates, where $\kappa''_\eta := \kappa_v + w_s + \kappa_\eta$,

$$\begin{aligned} \|\partial_{x, \zeta, \vartheta}^\nu f_q(x, \zeta, \vartheta, \zeta', \vartheta')\| &\leq C_{\nu, q} \langle \vartheta \rangle^{|\mu|} \langle \vartheta' \rangle^{m-p_{m, w_1}-q} \sum_{\mu' \leq \mu} \langle \zeta \rangle^{\kappa_v |\mu'| + w_0 + \kappa_\eta |\mu|} \\ &\quad \times \langle \zeta' \rangle^{w_1 + (\kappa_v + w_s) |\mu'| + \kappa_\eta |\mu| - (p_{m, w_1} + q) + w_s |\gamma|} \langle x \rangle^{\sigma |l|} \\ &\leq C'_{\nu, q} \langle x \rangle^{\sigma |l|} \langle \zeta \rangle^{w_0 + \kappa''_\eta |\mu|} \langle \vartheta \rangle^{|\mu|} \langle \vartheta' \rangle^{m-p_{m, w_1}-q} \langle \zeta' \rangle^{w_1 + \kappa''_\eta |\nu| - p_{m, w_1} - q}. \end{aligned}$$

If $k \in \mathbb{N}^*$, and if we set $q := q_k$ such that $w_1 + \kappa''_\eta k - p_{m, w_1} - q_k \leq -2n$, we see by applying the theorem of derivation under the integral sign that $S_{m, w}(f)$ is smooth and for any $3n$ -multi-index $\nu = (\alpha, \beta, \gamma)$ and $q \in \mathbb{N}^*$, after integrations by parts in ϑ' , with $\nu' := (\mu', \gamma)$,

$$\begin{aligned} \partial^\nu S_{m, w_1}(f)(x, \zeta, \vartheta) &= \sum_{\mu' \leq \mu} \sum_{|\omega| \leq |\mu'|} \binom{\mu}{\mu'} \vartheta^\omega \int_{\mathbb{R}^{2n}} e^{2\pi i(\langle \vartheta', \zeta' \rangle + \langle \vartheta, s_{x, \zeta}(\zeta') \rangle)} T_{\nu', \omega, s}(x, \zeta, \zeta') \\ &\quad {}^t M_{\vartheta'}^{p_{m, w_1} + q_{|\nu|} + q, \zeta'} \partial_{x, \zeta}^{\mu - \mu'} (f(1 - \chi_{\delta, \eta})) d\vartheta' d\zeta'. \end{aligned}$$

We note $g_q(x, \zeta, \zeta', \vartheta') := e^{2\pi i(\langle \vartheta', \zeta' \rangle)} T_{\nu', \omega, s}(x, \zeta, \zeta') {}^t M_{\vartheta'}^{p_{m, w_1} + q_{|\nu|} + q, \zeta'} \partial_{x, \zeta}^{\mu - \mu'} (f(1 - \chi_{\delta, \eta}))$. Using now Lemma 5.4.44, we get the estimates for any $p \in \mathbb{N}$,

$$\|(L_{\zeta'} h)^p g_q(x, \zeta, \zeta', \vartheta')\| \leq C_p \langle \zeta' \rangle^{2p\kappa_L} \langle \zeta \rangle^{2p\kappa_L} \langle \vartheta \rangle^{-2p} \sum_{k=1}^{N_p} \left\| \partial_{\zeta'}^{\beta^{k, p}} g_q(x, \zeta, \zeta', \vartheta') \right\|.$$

Thus, with Lemma 5.4.42 (i), we obtain with $k_1 := w_s + \kappa_v + \kappa_\eta + \kappa_L$,

$$\begin{aligned} \|(L_{\zeta'} h)^p g_q(x, \zeta, \zeta', \vartheta')\| &\leq C'_p \langle x \rangle^{\sigma |l|} \langle \zeta' \rangle^{w_1 + (2p + |\nu|)k_1 - p_{m, w_1} - q_{|\nu|} - q} \langle \vartheta \rangle^{-2p} \\ &\quad \langle \vartheta' \rangle^{2p + m - p_{m, w_1} - q_{|\nu|} - q} \langle \zeta \rangle^{(2p + |\mu|)k_1 + w_0} \sum_{|\tilde{\beta}| \leq 2p} \sum_{\mu' \leq \mu} \sum_{|\tilde{\delta}| = p_{m, w_1} + q_{|\nu|} + q} q_{\mu', \tilde{\beta}, \tilde{\delta}} (f(1 - \chi_{\delta, \eta})) 1_D(x, \zeta, \zeta') \end{aligned}$$

where $D := \{(x, \zeta, \zeta') \in \mathbb{R}^{2n} \mid \|\zeta'\| \geq \frac{1}{2} \delta \langle x \rangle^{\sigma \eta_1} \langle \zeta \rangle^{-\eta_2}\}$. If we now fix p such that $-N - 2 \leq -2p + |\mu| \leq -N$, we see that by taking q such that $A_q \leq -N/\eta_1 - |l|/\eta_1$ where $A_q := w_1 + (2p + |\nu|)k_1 - p_{m, w_1} - q_{|\nu|} - q + 2n$, and $2p + m - p_{m, w_1} - q_{|\nu|} - q \leq -2n$, we can successively integrate by parts in ζ' (p times) using the formula of Lemma 5.4.44. We obtain then the estimate for given constants $c_0, c_1, c_2 > 0$,

$$\begin{aligned} \|\partial^\nu S_{m, w_1}(f)(x, \zeta, \vartheta)\| &\leq C_{\nu, N} \langle x \rangle^{-\sigma N} \langle \zeta \rangle^{c_0 + c_1 N + c_2 |\mu|} \langle \vartheta \rangle^{-N} \\ &\quad \sum_{|\tilde{\beta}| \leq 2p} \sum_{\mu' \leq \mu} \sum_{|\tilde{\delta}| = p_{m, w_1} + q_{|\nu|} + q} q_{\mu', \tilde{\beta}, \tilde{\delta}} (f(1 - \chi_{\delta, \eta})) \end{aligned}$$

which yields the result.

(ii) This statement follows from (i) and Lemma 5.4.14 (ii). \square

Lemma 5.4.46. *Suppose (C_σ) .*

(i) *Defining for any $f \in \widetilde{\Pi}_{\sigma, \kappa, \varepsilon_1, z}^{l, w_0, w_1, m}$,*

$$\Pi(f) : (x, \zeta, \vartheta) \mapsto \int_{\mathbb{R}^{2n}} e^{2\pi i(\langle \vartheta', \zeta' \rangle + \langle \vartheta, \varphi_{x, \zeta}(\zeta') \rangle)} f(x, \zeta, \zeta', \vartheta' + L_{x, \zeta}(\vartheta)) \chi_{\delta, \eta}(x, \zeta, \zeta') d\zeta' d\vartheta',$$

there is δ, η , such that for any $N \geq |m|$, we have $\Pi(f) = \Pi_N(f) + \Pi_{R, N}(f)$ where $\Pi_N(f) = \sum_{0 \leq |\beta| \leq N} \frac{(i/2\pi)^{|\beta|}}{\beta!} f_{\beta, \varphi}$ and there is such that $\Pi_{R, N}(f)$ satisfies the estimates for any $3n$ -multi-index $\nu = (\mu, \gamma) \in \mathbb{N}^{2n} \times \mathbb{N}^n$,

$$\partial^\nu \Pi_{R, N}(f) = \mathcal{O}(\langle x \rangle^{\sigma(l - \varepsilon'_1(N+1))} \langle \zeta \rangle^{k_0 + k_1(N+1+|\mu|) + \varepsilon_v |\gamma|} \langle \vartheta \rangle^{m+|\mu| - (N+1)/2 + n})$$

where $\varepsilon'_1, k_0, k_1 > 0$.

(ii) *We have for any $3n$ -multi-index $\nu = (\mu, \gamma) \in \mathbb{N}^{2n} \times \mathbb{N}^n$,*

$$\partial^\nu \Pi(f) = \mathcal{O}(\langle x \rangle^{\sigma l} \langle \zeta \rangle^{k'_0 + k'_1 |\mu| + \varepsilon_v |\gamma|} \langle \vartheta \rangle^m)$$

where $k'_0, k'_1 > 0$. In particular, for any $u \in \mathcal{S}(\mathbb{R}^{2n}, L(E_z))$, the linear application $f \mapsto \langle \mathfrak{D}_{\mathbf{p}_\Gamma} \Pi(f), u \rangle$ is continuous.

Proof. (i) We proceed to a Taylor expansion of $\widetilde{f}(x, \zeta, \zeta', \vartheta', \vartheta) := f(x, \zeta, \zeta', \vartheta' + L_{x, \zeta}(\vartheta))$ in ϑ' around zero at order $N \in \mathbb{N}^*$, so that

$$\Pi(f) = \sum_{0 \leq |\beta| \leq N} \frac{1}{\beta!} I_\beta(f) + \sum_{|\beta| = N+1} \frac{N+1}{\beta!} R_{\beta, N}(f) =: \Pi_N(f) + \Pi_{R, N}(f)$$

where

$$I_\beta(f) = \int_{\mathbb{R}^{2n}} \vartheta'^\beta e^{2\pi i(\langle \vartheta', \zeta' \rangle + \langle \vartheta, \varphi_{x, \zeta}(\zeta') \rangle)} \partial^{0, 0, 0, \beta} f(x, \zeta, \zeta', L_{x, \zeta}(\vartheta)) \chi_{\delta, \eta}(x, \zeta, \zeta') d\zeta' d\vartheta',$$

$$R_{\beta, N}(f) = \int_{\mathbb{R}^{2n}} \vartheta'^\beta e^{2\pi i(\langle \vartheta', \zeta' \rangle + \langle \vartheta, \varphi_{x, \zeta}(\zeta') \rangle)} r_{\beta, N, f}(x, \zeta, \zeta', \vartheta', \vartheta) d\zeta' d\vartheta',$$

and $r_{\beta, N, f} := \int_0^1 (1-t)^N \partial^{0, 0, 0, \beta} f_\chi(x, \zeta, \zeta', t\vartheta' + L_{x, \zeta}(\vartheta)) dt$, $f_\chi := f \chi_{\delta, \eta} \in \widetilde{\Pi}_{\sigma, \kappa_\eta, z}^{l, w_0, w_1, m}$. By integration by parts in ζ' in the integrals $I_\beta(f)$, we get

$$\Pi_N(f) = \sum_{0 \leq |\beta| \leq N} \frac{(i/2\pi)^{|\beta|}}{\beta!} \partial_{\zeta'}^\beta (e^{2\pi i \langle \vartheta, \varphi_{x, \zeta}(\zeta') \rangle} \partial^{0, 0, 0, \beta} f(x, \zeta, \zeta', L_{x, \zeta}(\vartheta)))_{\zeta'=0} = \sum_{0 \leq |\beta| \leq N} \frac{(i/2\pi)^{|\beta|}}{\beta!} f_{\beta, \varphi}.$$

Using integration by parts in ζ' , we obtain $R_{\beta, N, f} = (i/2\pi)^{|\beta|} I_f$, where for any $p \in \mathbb{N}$,

$$I_f(x, \zeta, \vartheta) := \int_{\mathbb{R}^{2n}} e^{2\pi i \langle \vartheta', \zeta' \rangle} \partial_{\zeta'}^\beta G(x, \zeta, \zeta', \vartheta', \vartheta) d\zeta' d\vartheta',$$

$$G(x, \zeta, \zeta', \vartheta', \vartheta) := e^{2\pi i \langle \vartheta, \varphi_{x, \zeta}(\zeta') \rangle} r_{\beta, N, f}(x, \zeta, \zeta', \vartheta', \vartheta).$$

Using integration by parts in ζ' and $e^{2\pi i \langle \vartheta', \zeta' \rangle} = \langle \vartheta' \rangle^{-2p} L_{\zeta'}^p e^{2\pi i \langle \vartheta', \zeta' \rangle}$, we check that I_f is smooth on \mathbb{R}^{3n} and if ν is a $3n$ -multi-index, we see that $\partial^\nu I_f$ is a linear combination of terms of the form

$$J_f := \vartheta^{\widetilde{\omega}} \int_{\mathbb{R}^{2n}} e^{2\pi i(\langle \vartheta', \zeta' \rangle + \langle \vartheta, \varphi_{x, \zeta}(\zeta') \rangle)} \partial_{\zeta'}^{\beta_1} T_{\nu', \widetilde{\omega}, \varphi} P_{\beta^2, \varphi} \partial_{x, \zeta, \vartheta}^{\nu - \nu'} \partial_{\zeta'}^{\beta_3} r_{\beta, N, f} d\zeta' d\vartheta'$$

where $|\tilde{\omega}| \leq |\mu'|$, $\nu' \leq \nu$, $\sum \beta^i = \beta$, $|\beta| = N + 1$. We now cut the integral J_f in two parts $J_\chi + J_{1-\chi}$, where the cut-off function $\chi_\varepsilon(\vartheta, \vartheta')$ appears in J_χ .

Analysis of J_χ

Using Lemma 5.4.42 (ii) and integration by parts in ζ' , we see that J_χ is a linear combination of terms of the form

$$J_{\chi, \omega} = \vartheta^{\tilde{\omega}} \vartheta^\omega \int_{\mathbb{R}^{2n}} e^{2\pi i (\langle \vartheta', \zeta' \rangle + \langle \vartheta, \varphi_{x, \zeta}(\zeta') \rangle)} \langle \zeta' \rangle^{-2p} t_{\omega, \lambda} \partial_{\zeta'}^{\beta^1} T_{\nu', \tilde{\omega}, \varphi} \partial_{\vartheta'}^{\lambda'} \partial_{x, \zeta, \vartheta}^{\nu - \nu'} \partial^{\beta^3} r_{\beta, N, f} \partial^{\lambda + \rho - \lambda'} \chi_\varepsilon d\zeta' d\vartheta'$$

where $p \in \mathbb{N}$, $|\rho| \leq 2p$, $|\omega| \leq |\beta^2|$, $(2|\omega| - |\beta^2|)_+ \leq |\lambda| \leq |\omega|$, $\lambda' \leq \lambda + \rho$. We now fix ε such that $\varepsilon < c/2$ where c is a constant such that $c\langle \vartheta \rangle \leq \langle L_{x, \zeta}(\vartheta) \rangle$. Thus, in the domain of integration of $J_{\chi, \omega}$, we have for any $t \in [0, 1]$, $\langle t\vartheta' + L_{x, \zeta}(\vartheta) \rangle \geq c_1 \langle \vartheta \rangle$ for a $c_1 > 0$. As a consequence, we obtain the following estimate:

$$\left\| \partial_{\vartheta'}^{\lambda'} \partial_{x, \zeta, \vartheta}^{\nu - \nu'} \partial_{\zeta'}^{\beta^3} r_{\beta, N, f} \right\| \leq C \langle x \rangle^{\sigma(l - \varepsilon_1 |\beta^3|)} \langle \zeta \rangle^{(\kappa_v + \kappa_\eta) |\mu - \mu'| + w_0 + \kappa_\eta |\beta^3|} \langle \zeta' \rangle^{w_1 + \kappa_\eta (|\mu - \mu'| + |\beta^3|)} \langle \vartheta \rangle^{|\mu - \mu'| + m - |\beta| - |\lambda'|}.$$

We also deduce from Lemma 5.4.42 the estimate

$$|t_{\omega, \lambda} \partial_{\zeta'}^{\beta^1} T_{\nu', \tilde{\omega}, \varphi}| \leq C' \langle x \rangle^{-\sigma(|\mu'| + (\varepsilon/2)|\beta^1 + \beta^2|)} \langle \zeta \rangle^{2\varepsilon_v |\beta^1 + \beta^2| + (\kappa_v + \varepsilon_v) |\mu| + \varepsilon_v |\gamma|} \langle \zeta' \rangle^{c_1(N+1) + c_2 |\nu'|}.$$

As a consequence, by taking p sufficiently big, the integrand $j(x, \zeta, \zeta', \vartheta, \vartheta')$ of $J_{\chi, \omega}$ satisfies the estimate, for a $\varepsilon'_1 > 0$ and a $k_1 > 0$,

$$\|j\| \leq C'' \langle x \rangle^{\sigma(l - \varepsilon'_1(N+1))} \langle \zeta \rangle^{w_0 + k_1(N+1+|\mu|) + \varepsilon_v |\gamma|} \langle \zeta' \rangle^{-2n} \langle \vartheta \rangle^{m+|\mu| - (N+1)/2} 1_{D_\varepsilon}(\vartheta, \vartheta')$$

where D_ε is the set of (ϑ, ϑ') in \mathbb{R}^{2n} such that $\|\vartheta'\| \leq \varepsilon \langle \vartheta \rangle$. We deduce finally that for any $\nu \in \mathbb{N}^{3n}$,

$$J_\chi = \mathcal{O}(\langle x \rangle^{\sigma(l - \varepsilon'_1(N+1))} \langle \zeta \rangle^{w_0 + k_1(N+1+|\mu|) + \varepsilon_v |\gamma|} \langle \vartheta \rangle^{m+|\mu| - (N+1)/2 + n}).$$

Analysis of $J_{1-\chi}$

We set $\omega := \langle \zeta', \vartheta' \rangle + \langle \vartheta, \varphi_{x, \zeta}(\zeta') \rangle$. By Lemma 5.4.39 (i), we have $\sum_i \left\| \partial_{\zeta'_i} \varphi_{x, \zeta}(\zeta') \right\| \leq C \langle x \rangle^{-\sigma \varepsilon_v} \langle \zeta \rangle^{c_1} \langle \zeta' \rangle^{c_2}$ for $C, c_1, c_2 > 0$. The presence of $\chi_{\delta, \eta}$ in the integrand of $J_{1-\chi}$ allows to use the estimate $\langle \zeta' \rangle \leq \sqrt{2} \delta \langle x \rangle^{\sigma \eta_1} \langle \zeta \rangle^{-\eta_2}$, so that $\sum_i \left\| \partial_{\zeta'_i} \varphi_{x, \zeta}(\zeta') \right\| \leq C 2^{c_2/2} \delta^{c_2}$ by taking $\eta_1 \leq \varepsilon_v / c_2$ and $\eta_2 \geq c_1 / c_2$. As a consequence, we obtain the following estimate in the domain of integration of $J_{1-\chi}$,

$$|\nabla_{\zeta'} \omega|^2 \geq \|\vartheta'\|^2 \left(1 - \frac{4}{\varepsilon} C 2^{c_2/2} \delta^{c_2}\right).$$

We now fix δ such that $\frac{4}{\varepsilon} C 2^{c_2/2} \delta^{c_2} < 1$ so that there is $k > 0$ such that $|\nabla_{\zeta'} \omega| \geq k \|\vartheta'\|$. Noting $U_{\zeta'} := (2\pi i |\nabla_{\zeta'} \omega|^2)^{-1} \sum_i (\partial_{\zeta'_i} \omega) \partial_{\zeta'_i}$ we have (see for instance [119]) $U_{\zeta'} e^{2\pi i \omega} = e^{2\pi i \omega}$ and

$$({}^t U_{\zeta'})^r = \frac{1}{|\nabla_{\zeta'} \omega|^{4r}} \sum_{|\rho| \leq r} P_{\rho, r}^\omega \partial_{\zeta'}^\rho$$

where $P_{\rho, r}^\omega$ is a linear combination of terms of the form $(\nabla_{\zeta'} \omega)^\pi \partial_{\zeta'}^{\delta^1} \omega \cdots \partial_{\zeta'}^{\delta^r} \omega$, with $|\pi| = 2r$, $|\delta^i| > 0$ and $\sum_{j=1}^r |\delta^j| + |\rho| = 2r$. We thus obtain after integration by parts in ζ' , for any $r \in \mathbb{N}^*$, that $J_{1-\chi}$ is a linear combination of integrals of the form

$$\vartheta^{\tilde{\omega} + \hat{\omega}} \int_{\mathbb{R}^{2n}} e^{2\pi i \omega} ({}^t U_{\zeta'})^r (\partial_{\zeta'}^{\beta^1} T_{\nu', \tilde{\omega}, \varphi} P_{\tilde{\omega}, \beta^2, \varphi} \partial_{x, \zeta, \vartheta}^{\nu - \nu'} \partial_{\zeta'}^{\beta^3} r_{\beta, N, f}) (1 - \chi_\varepsilon) d\zeta' d\vartheta'$$

where $|\widehat{\omega}| \leq |\beta^2|$. We noted $P_{\beta^2, \varphi} =: \sum_{\widehat{\omega}} P_{\widehat{\omega}, \beta^2, \varphi} \vartheta^{\widehat{\omega}}$. By Lemma 5.4.42 (ii), we see that $P_{\widehat{\omega}, \beta^2, \varphi} \in \mathcal{O}^{-\varepsilon_v |\beta^2|/2, 2\varepsilon_v |\beta^2|, (w'_s+1)|\beta^2|}_{\sigma, \kappa_v, \varepsilon_v, \varepsilon_v, 2|\beta^2|}$. Let us note $\widetilde{T} := \partial_{\zeta'}^{\beta^1} T_{\nu', \widetilde{\omega}, \varphi} P_{\widehat{\omega}, \beta^2, \varphi}$. Lemma 5.4.42 (i) yields $\widetilde{T} \in \mathcal{O}^{-(\varepsilon_v/2)|\beta^1+\beta^2|, c_0(|\mu|+N)+\varepsilon_v|\gamma|, c_0(|\nu|+N)}_{\sigma, \kappa_v, \varepsilon_v, \varepsilon_v, 2(|\nu|+N)}(\mathbb{R})$ for a constant $c_0 > 0$. With our choice of the parameters η_1 and η_2 , we also have the following estimate, valid in the domain of integration of $J_{1-\chi}$,

$$\partial_{\vartheta'}^\lambda \partial_{\zeta'}^{\gamma+e_i} \omega = \mathcal{O}(\langle \zeta \rangle^{\varepsilon_v |\gamma|} \langle \zeta' \rangle^{\kappa_v |\gamma|} \langle \vartheta' \rangle^{1-|\lambda|}).$$

In particular, noting $\mathcal{O}_{\kappa_v}^{l, m}$ the space of smooth functions f such that for any n -multi-indices λ, γ , $\partial_{\vartheta'}^\lambda \partial_{\zeta'}^\gamma f = \mathcal{O}((\langle \zeta \rangle \langle \zeta' \rangle)^{l+\kappa_v |\gamma|} \langle \vartheta' \rangle^m)$, we see that $|\nabla_{\zeta'} \omega|^2 \in \mathcal{O}_{\kappa_v}^{0, 2}$, and for any $\lambda \in \mathbb{N}^n$, $\partial_{\vartheta'}^\lambda |\nabla_{\zeta'} \omega|^{-4r} = \mathcal{O}(\langle \vartheta' \rangle^{-4r})$. Moreover, each term $P_{\rho, r}^\omega$ is in $\mathcal{O}_{\kappa_v}^{\kappa_v r, 3r}$ so that finally, for any $\lambda \in \mathbb{N}^n$

$$\partial_{\vartheta'}^\lambda \frac{P_{\rho, r}^\omega}{|\nabla_{\zeta'} \omega|^{4r}} = \mathcal{O}((\langle \zeta \rangle \langle \zeta' \rangle)^{\kappa_v r} \langle \vartheta' \rangle^{-r}).$$

We easily check that if $r \geq 2n$, then $h := ({}^t U_{\zeta'})^r (\partial_{\zeta'}^{\beta^1} \widetilde{T} \partial_{x, \zeta, \vartheta}^{\nu-\nu'} \partial_{\zeta'}^{\beta^3} r_{\beta, N, f}) (1 - \chi_\varepsilon)$ satisfies the estimates for any $q \in \mathbb{N}$, $\|L_{\vartheta'}^q h\| \leq C_{x, \zeta, \zeta', \vartheta, q} \langle \vartheta' \rangle^{-2n}$. As a consequence, we can permute the integration $d\zeta' d\vartheta' \rightarrow d\vartheta' d\zeta'$ and successively integrate by parts in ϑ' , so that finally $J_{1-\chi}$ is a linear combination of terms of the form

$$\vartheta^{\widetilde{\omega}+\widehat{\omega}} \int_{\mathbb{R}^{2n}} e^{2\pi i \omega \langle \zeta' \rangle - 2q} \partial_{\vartheta'}^{\lambda^1} \frac{P_{\rho, r}^\omega}{|\nabla_{\zeta'} \omega|^{4r}} \partial_{\zeta'}^{\rho^1} \widetilde{T} \partial_{\vartheta'}^{\lambda^2} \partial_{x, \zeta, \vartheta}^{\nu-\nu'} \partial_{\zeta'}^{\beta^3+\rho^2} r_{\beta, N, f} \partial_{\vartheta'}^{\lambda^3} (1 - \chi_\varepsilon) d\vartheta' d\zeta'$$

where $\sum_i \lambda^i = \lambda$, $|\lambda| \leq 2q$, $\sum_i \rho^i = \rho$, $|\rho| \leq r$. We also have the following estimate for $c'_0, c'_1 > 0$,

$$\partial_{\vartheta'}^{\lambda^2} \partial_{x, \zeta, \vartheta}^{\nu-\nu'} \partial_{\zeta'}^{\beta^3+\rho^2} r_{\beta, N, f} = \mathcal{O}(\langle x \rangle^{\sigma(l-|\beta^3|)} (\langle \zeta \rangle \langle \zeta' \rangle)^{c'_0+c'_1(|\mu-\mu'|+|\beta^3|+|\rho^2|)}).$$

With Lemma 5.4.43 (iv) we now see that the integrand j' of the previous integral is estimated by

$$\|j'\| \leq C \langle \vartheta' \rangle^{-r+|\mu|+N+1} \langle x \rangle^{\sigma(l-\varepsilon'_1(N+1))} \langle \zeta \rangle^{k_0+k_1N+k_2r+k_3|\mu|+\varepsilon_v|\gamma|} \langle \zeta' \rangle^{-2q+k_0+k_1N+k_2r+k_3|\nu|}$$

for constants $k_0, k_1, k_2, k_3 > 0$. If we now fix $r \geq 2n$ such that $-r + |\mu| + N + 1 + 2n = m + |\mu| - (N + 1) + n$, and q such that $-2q + k_0 + k_1N + k_2r + k_3|\nu| \leq -2n$ we finally obtain the estimate $\nu \in \mathbb{N}^{3n}$,

$$J_{1-\chi} = \mathcal{O}(\langle x \rangle^{\sigma(l-\varepsilon'_1(N+1))} \langle \zeta \rangle^{k'_0+k'_1(N+1+|\mu|)+\varepsilon_v|\gamma|} \langle \vartheta \rangle^{m+|\mu|-(N+1)+n}).$$

The result follows now from this estimate and the one obtained for J_χ .

(ii) The estimate is obtained by applying (i) and $N + 1 = \max\{2(n + |\mu|), |m|\}$. The second statement is then a consequence of Lemma 5.4.14 (ii). \square

Theorem 5.4.47. *If (C_σ) holds, Ψ_σ^∞ is a $*$ -subalgebra of $\mathfrak{R}(\mathcal{S})$. Moreover, if $A \in \Psi_\sigma^{l', m'}$ and $B \in \Psi_\sigma^{l, m}$, then $AB \in \Psi_\sigma^{l+l', m+m'}$ with the following asymptotic expansion of the normal symbol of AB , in a frame (z, \mathbf{b}) :*

$$\sigma_0(AB)_{z, \mathbf{b}} \sim \sum_{\beta, \gamma \in \mathbb{N}^n} c_\beta c_\gamma \partial_{\zeta, \vartheta}^{\gamma, \gamma} (a(x, \vartheta) \partial_{\zeta'}^\beta (e^{2\pi i \langle \vartheta, \varphi_{x, \zeta}(\zeta') \rangle} (\partial_{\vartheta'}^\beta f_b)(x, \zeta, \zeta', L_{x, \zeta}(\vartheta)))_{\zeta'=0} \tau_{x, \zeta}^{-1})_{\zeta=0}$$

where $a := \sigma_0(A)_{z, \mathbf{b}}$, $b := \sigma_0(B)_{z, \mathbf{b}}$, $c_\beta := (i/2\pi)^{|\beta|}/\beta!$ and

$$f_b(x, \zeta, \zeta', \vartheta') := \tau_{x, r_{x, \zeta}(\zeta')} b \circ \widetilde{\Xi}(x, \zeta, \zeta', \vartheta') \tau_{x, \zeta, \zeta', q_{x, \zeta}(\zeta')} |J(R)|(x, \zeta, \zeta') |\det(P_{-1, \psi(x, \zeta), \zeta'}^{z, \mathbf{b}})^{-1}|.$$

Proof. We fix a frame (z, \mathbf{b}) . We note K_{AB} the kernel of the operator AB . As a consequence of Proposition 5.4.32 we have for any $u, v \in \mathcal{S}(\mathbb{R}^n, E_z)$, $\langle (K_{AB})_{z, \mathbf{b}}, u \otimes \bar{v} \rangle = (A_{z, \mathbf{b}}(\mu^{-1} B_{z, \mathbf{b}}(v)) | u)$. We shall note $g := A_{z, \mathbf{b}}(\mu^{-1} B_{z, \mathbf{b}}(v))$. A computation shows that for any $\mathbf{x} \in \mathbb{R}^n$, $g(\mathbf{x}) = \int_{\mathbb{R}^n} \mu a(\mathbf{x}, \vartheta) \tilde{b}(\mathbf{x}, \vartheta) d\vartheta$, and

$$\tilde{b}(\mathbf{x}, \vartheta) := \int_{\mathbb{R}^{3n}} e^{2\pi i(\langle \vartheta, \zeta \rangle + \langle \vartheta', \zeta' \rangle)} \tau_{\mathbf{x}, \zeta} b(\psi(\mathbf{x}, \zeta), \vartheta') \tau_{\psi(\mathbf{x}, \zeta), \zeta'} v(\mathbf{x}^{\zeta, \zeta'}) d\zeta' d\vartheta' d\zeta.$$

We suppose at first that $b \in S_{\sigma, z}^{l, -2n}$. Since $\zeta' \mapsto v(\mathbf{x}^{\zeta, \zeta'}) \in \mathcal{S}(\mathbb{R}^n, E_z)$, we can permute the order integration $d\zeta' d\vartheta' \mapsto d\vartheta' d\zeta'$ in $\tilde{b}(\mathbf{x}, \vartheta)$. Thus, after integrations by parts in ϑ' , we get for any $p \in \mathbb{N}^*$,

$$\tilde{b}(\mathbf{x}, \vartheta) = \int_{\mathbb{R}^{2n}} e^{2\pi i \langle \vartheta, \zeta \rangle} \tau_{\mathbf{x}, \zeta} \left(\int_{\mathbb{R}^n} e^{2\pi i \langle \vartheta', \zeta' \rangle} \langle \zeta' \rangle^{-2p} (L_{\vartheta'}^p b)(\psi(\mathbf{x}, \zeta), \vartheta') d\vartheta' \right) \tau_{\psi(\mathbf{x}, \zeta), \zeta'} v(\mathbf{x}^{\zeta, \zeta'}) d\zeta' d\zeta.$$

With the estimate $\langle \mathbf{x}^{\zeta, \zeta'} \rangle \geq c \langle \zeta \rangle \langle \mathbf{x} \rangle^{-1} \langle \zeta' \rangle^{-1}$ for a $c > 0$, we see that for any $N \in \mathbb{N}$, $\|v(\mathbf{x}^{\zeta, \zeta'})\| \leq c_N q_{0, N}(v) \langle \mathbf{x} \rangle^N \langle \zeta' \rangle^N \langle \zeta \rangle^{-N}$. As a consequence, we get the following estimates for the integrands b_p of $\tilde{b}(\mathbf{x}, \vartheta)$: for any $\mathbf{x}, \zeta, \zeta', \vartheta, \vartheta'$, any $p \in \mathbb{N}^*$ and any $N \in \mathbb{N}^*$, $\|b_p(\mathbf{x}, \zeta, \zeta', \vartheta, \vartheta')\| \leq C_{p, N} \langle \zeta' \rangle^{N-2p} \langle \mathbf{x} \rangle^{\sigma|l|+N} \langle \zeta \rangle^{\sigma|l|-N} \langle \vartheta' \rangle^{-2n}$. Taking N such that $\sigma|l| - N \leq -2n$ and then taking p such that $N - 2p \leq -2n$, we see that $(\vartheta', \zeta', \zeta) \mapsto b_p(\mathbf{x}, \zeta, \zeta', \vartheta', \vartheta)$ is absolutely integrable and we can thus apply the following change of variable $(\zeta, \zeta', \vartheta') \mapsto (R_{\mathbf{x}}(\zeta, \zeta'), \vartheta')$ to $\tilde{b}(\mathbf{x}, \vartheta)$. After reversing the integration by parts in ϑ' and applying the change of variable $\vartheta' = -\tilde{P}_{-1, \psi(\mathbf{x}, \zeta), \zeta'}^{z, \mathbf{b}}(\vartheta'')$, we get

$$\tilde{b}(\mathbf{x}, \vartheta) = \int_{\mathbb{R}^{3n}} e^{2\pi i(\langle \vartheta, r_{\mathbf{x}, \zeta}(\zeta') \rangle + \langle \vartheta', \zeta' \rangle)} f_b(\mathbf{x}, \zeta, \zeta', \vartheta') v(\psi(\mathbf{x}, \zeta)) d\vartheta' d\zeta' d\zeta.$$

By Lemma 5.4.40 (ii) and (iii), Lemma 5.4.38 (iii) and (iv) and Lemma 5.4.39 (iii), we see that $f_b \in \tilde{\Pi}_{\sigma, \kappa, \varepsilon_1, z}^{l, w_l, w_l, m}$ for a $(w_l, \kappa) \in \mathbb{R}_+^2$ and $\varepsilon_1 > 0$, and the linear application $b \mapsto f_b$ is continuous on any symbol space $S_{\sigma, z}^{l, m}$ into $\tilde{\Pi}_{\sigma, \kappa, \varepsilon_1, z}^{l, w_l, w_l, m}$. We have $g(\mathbf{x}) = \int_{\mathbb{R}^n} e^{2\pi i \langle \zeta, \vartheta \rangle} \mu a(\mathbf{x}, \vartheta) c_b(\mathbf{x}, \zeta, \vartheta) v(\psi(\mathbf{x}, \zeta)) d\zeta d\vartheta$ and $\langle (K_{AB})_{z, \mathbf{b}}, u \otimes \bar{v} \rangle = \langle \mathfrak{Op}_{\Gamma_{0, z, \mathbf{b}}} (d_b), u \otimes \bar{v} \rangle$ where $d_b(\mathbf{x}, \zeta, \vartheta) := \mu a(\mathbf{x}, \vartheta) c_b(\mathbf{x}, \zeta, \vartheta) \tau^{-1}(\mathbf{x}, \zeta)$ and

$$c_b(\mathbf{x}, \zeta, \vartheta) := \int_{\mathbb{R}^{2n}} e^{2\pi i(\langle \vartheta, s_{\mathbf{x}, \zeta}(\zeta') \rangle + \langle \vartheta', \zeta' \rangle)} f_b(\mathbf{x}, \zeta, \zeta', \vartheta') d\vartheta' d\zeta'.$$

Using now the cut-off function $(\mathbf{x}, \zeta, \zeta') \mapsto \chi_{\delta, \eta}(\mathbf{x}, \zeta, \zeta')$ we see that

$$c_b(\mathbf{x}, \zeta, \vartheta) = \Pi(f_b)(\mathbf{x}, \zeta, \vartheta) + S_{m, w_l}(f_b)(\mathbf{x}, \zeta, \vartheta).$$

For this equality, we used the formula of Lemma 5.26 and integration by parts in ϑ' in the integral $\int_{\mathbb{R}^{2n}} e^{2\pi i(\langle \vartheta, s_{\mathbf{x}, \zeta}(\zeta') \rangle + \langle \vartheta', \zeta' \rangle)} f_b(\mathbf{x}, \zeta, \zeta', \vartheta') (1 - \chi_{\delta, \eta}(\mathbf{x}, \zeta, \zeta')) d\vartheta' d\zeta'$, which are authorized since $b \in S_{\sigma, z}^{l, -2n}$ by hypothesis. In $\int_{\mathbb{R}^{2n}} e^{2\pi i(\langle \vartheta, s_{\mathbf{x}, \zeta}(\zeta') \rangle + \langle \vartheta', \zeta' \rangle)} f_b(\mathbf{x}, \zeta, \zeta', \vartheta') \chi_{\delta, \eta}(\mathbf{x}, \zeta, \zeta') d\vartheta' d\zeta'$, we translated the ϑ' variable by $-L_{\mathbf{x}, \zeta}(\vartheta)$ and permuted the order of integration $d\vartheta' d\zeta' \rightarrow d\zeta' d\vartheta'$, which is legal since $b \in S_{\sigma, z}^{l, -2n}$ and $\zeta' \mapsto \chi(\mathbf{x}, \zeta, \zeta')$ is of compact support. We deduce from Lemma 5.4.45 (ii) and Lemma 5.4.46 (ii) that $b \mapsto \langle \mathfrak{Op}_{\Gamma_{0, z, \mathbf{b}}} (d_b), u \otimes \bar{v} \rangle$ is continuous on $S_{\sigma, z}^{l, m}$, and thus, by the density result of Lemma 5.4.6, we have the equality $\langle (K_{AB})_{z, \mathbf{b}}, u \otimes \bar{v} \rangle = \langle \mathfrak{Op}_{\Gamma_{0, z, \mathbf{b}}} (d_b), u \otimes \bar{v} \rangle$ even when the hypothesis $b \in S_{\sigma, z}^{l, -2n}$ does not hold.

Let us recall the linear map $s : a \mapsto s(a)$ given in Lemma 5.4.21 (ii) (for $\Gamma = \Gamma_{0,z,b}$) which is such that $\mathfrak{Op}_{\Gamma_{0,z,b}}(f) = \mathfrak{Op}_{\Gamma_{0,z,b}}(s(f))$ for any $f \in \Pi_{\sigma,\kappa,z}^{l,w,m}$. We define $f_{a,b,\beta} := \mu a(f_b)_{\beta,\varphi} \tau^{-1}$, $r_N := \mu a \Pi_{R,N}(f_b) \tau^{-1}$, $s_0 := \mu a S_{m,w_l}(f_b) \tau^{-1}$. We now consider a symbol $s_{a,b}$ such that

$$s_{a,b} \sim \sum_{\beta \in \mathbb{N}^n} \frac{(i/2\pi)^{|\beta|}}{\beta!} s(f_{a,b,\beta}).$$

Such a symbol exists since by Lemma 5.4.42 (iii), $s(f_{a,b,\beta}) \in S_{\sigma,z}^{l+l'-\varepsilon'_1|\beta|,m+m'-|\beta|/2}$. By Lemma 5.4.46 (i), we have for any $N \geq |m|$, $u_N := s(\mu a \Pi_N(f_b) \tau^{-1}) - s_{a,b} \in S_{\sigma,z}^{l+l'-\varepsilon'_1(N+1),m+m'-(N+1)/2}$. Thus, noting $S_0 := \mathfrak{Op}_{\Gamma_{0,z,b}}(s_0)$, which is in $\mathfrak{Op}_{\Gamma_{0,z,b}}(S_{\sigma,z}^{-\infty})$ by Lemma 5.4.45, $R_N := \mathfrak{Op}_{\Gamma_{0,z,b}}(r_N)$ and $U_N := \mathfrak{Op}_{\Gamma_{0,z,b}}(u_N)$ we have

$$\begin{aligned} (K_{AB})_{z,b} &= \mathfrak{Op}_{\Gamma_{0,z,b}}(d_b) = \mathfrak{Op}_{\Gamma_{0,z,b}}(s(\mu a \Pi_N(f_b) \tau^{-1})) + R_N + S_0 \\ &= \mathfrak{Op}_{\Gamma_{0,z,b}}(s_{a,b}) + U_N + R_N + S_0. \end{aligned}$$

Lemma 5.4.18 and Lemma 5.4.46 (i) now implies that the kernel $U_N + R_N$ (which independent of N) is in $\mathfrak{Op}_{\Gamma_{0,z,b}}(S_{\sigma,z}^{-\infty})$. As a consequence, $(K_{AB})_{z,b} = \mathfrak{Op}_{\Gamma_{0,z,b}}(s_{a,b} + r)$ where $r \in S_{\sigma,z}^{-\infty}$ and the symbol product asymptotic formula is entailed by Lemma 5.4.21 (ii). \square

5.5 Examples

In order to be able to apply the previous results about the pseudodifferential and symbolic calculi on some concrete cases, we shall see in this section examples of exponential manifolds and associated linearizations that satisfy the hypothesis S_σ -bounded geometry. The Euclidean space \mathbb{R}^n seen as exponential manifold, has its own exponential map $\psi := \exp(x, \xi) \mapsto x + \xi$ as a S_1 -linearization, leading to the usual pseudodifferential SG calculus (if $\sigma = 1$) or standard (if $\sigma = 0$) pseudodifferential calculus on \mathbb{R}^n . However, we can define other kinds of linearization, leading to new kind of pseudodifferential and symbol calculi, with a non-bilinear linearization map. We will see in particular that we can construct on the flat \mathbb{R}^n , a family of S_σ -linearizations that generalize the case of the flat euclidian geometry, and we obtain a extension of the normal ($\lambda = 0$) and antinormal ($\lambda = 1$) quantization on \mathbb{R}^n .

We will also prove that the 2-dimensional hyperbolic space, which is a Cartan–Hadamard manifold (and thus an exponential Riemannian manifold) has S_1 -bounded geometry. This allows to define a global Fourier transform, Schwartz spaces $\mathcal{S}(\mathbb{H})$, $\mathcal{S}(T^*\mathbb{H})$, $\mathcal{S}(T\mathbb{H})$, $\mathcal{B}(\mathbb{H})$ and the space of symbols $S_1^{l,m}(T^*\mathbb{H})$. As a consequence, we can define in an intrinsic way a global complete pseudodifferential calculus on \mathbb{H} , if one chose a fixed S_σ -linearization ψ on $T\mathbb{H}$. There are many possible linearizations, for instance one can take ψ such that in a frame (z, \mathbf{b}) $\psi_z^{\mathbf{b}}$ is the standard linearization $x + \xi$ of \mathbb{R}^n .

5.5.1 A family of S_σ -linearizations on the euclidean space

Recall that $G_\sigma^\times(\mathbb{R}^n)$ ($0 \leq \sigma \leq 1$) is defined as the subgroup of diffeomorphisms s on \mathbb{R}^n such that for any n -multi-index $\alpha \neq 0$, there are $C_\alpha, C'_\alpha > 0$, such that for any $x \in \mathbb{R}^n$, $\|\partial^\alpha s(x)\| \leq C_\alpha \langle x \rangle^{\sigma(1-|\alpha|)}$ and $\|\partial^\alpha s^{-1}(x)\| \leq C'_\alpha \langle x \rangle^{\sigma(1-|\alpha|)}$. $G_\sigma^\times(\mathbb{R}^n)$ contains $GL_n(\mathbb{R})$ and the translations $T_v := w \mapsto v + w$.

We fix $\eta \in]0, 1[$ such that for any matrix $A \in \mathcal{M}_n(\mathbb{R})$ such that $\|A\|_1 \leq \eta$, we have $\det(I_n + A) \geq \frac{1}{2}$, where $\|A\|_1 := \max_{i,j} |A_{i,j}|$. Taking now $h \in G_0(\mathbb{R}^n, \mathbb{R}^n)$ such that for any $1 \leq i, j \leq n$, $|\partial_j h^i| \leq \eta/16$, and $g(x) := h(x) - h(0) - dh_0(x)$ we see that $s := \text{Id} + g$ is a diffeomorphism on \mathbb{R}^n which belongs to $G_0^\times(\mathbb{R}^n)$, satisfying $s(0) = 0$ and $ds_0 = \text{Id}$.

We set, for $\sigma \in [0, 1]$,

$$\psi(x, \xi) := x + \xi + \langle x \rangle^\sigma g\left(\frac{\xi}{\langle x \rangle^\sigma}\right) = x + \langle x \rangle^\sigma s\left(\frac{\xi}{\langle x \rangle^\sigma}\right).$$

We obtain the following

Proposition 5.5.1. $(\mathbb{R}^n, +, d\lambda, \psi)$ has a S_σ -bounded geometry and satisfies (C_σ) (see Definition 5.4.37).

Proof. A computation shows that $\psi \in H_\sigma(\mathbb{R}^n)$ and $\psi(x, \xi) = \mathcal{O}(\langle x \rangle \langle \xi \rangle)$. We have $\bar{\psi}(x, y) = \langle x \rangle^\sigma s^{-1}\left(\frac{y-x}{\langle x \rangle^\sigma}\right)$, and thus $\bar{\psi} \in \mathcal{O}_M(\mathbb{R}^{2n}, \mathbb{R}^n)$. Noting $\widehat{g} := g \circ (g + \text{Id})^{-1} \circ -\text{Id} \in G_0(\mathbb{R}^n)$, we also have

$$\begin{aligned} \Upsilon_{1,T}(x, \xi) &= \xi + \langle x \rangle^\sigma g\left(\frac{\xi}{\langle x \rangle^\sigma}\right) + \langle \psi(x, \xi) \rangle^\sigma \widehat{g}\left(\langle \psi(x, \xi) \rangle^{-\sigma} \langle x \rangle^\sigma s\left(\frac{\xi}{\langle x \rangle^\sigma}\right)\right) \\ &= (\text{Id} + V_{x,\xi} + W_{x,\xi})(\xi) \end{aligned}$$

where $V_{x,\xi} := [\int_0^1 \partial_j v_x^i(t\xi) dt]_{i,j}$, $W_{x,\xi} := [\int_0^1 \partial_j w_{x,\xi}^i(t\xi) dt]_{i,j}$, and $v_x := M_x \circ g \circ M_x^{-1}$, $w_{x,\xi} := M_{\psi(x,\xi)} \circ \widehat{g} \circ M_{\psi(x,\xi)}^{-1} \circ M_x \circ s \circ M_x^{-1}$, M_x being the multiplication by $\langle x \rangle^\sigma$. We get $dv_x = dg \circ M_x^{-1}$ and $dw_{x,\xi} = d\widehat{g} \circ (M_{\psi(x,\xi)}^{-1} \circ M_x \circ s \circ M_x^{-1}) ds \circ M_x^{-1}$. and thus, after computations we check that $V_{x,\xi}$ and $W_{x,\xi}$ are in E_σ^0 . Moreover, we have $\|V_{x,\xi}\|_1 \leq \eta/2$ and $\|W_{x,\xi}\|_1 \leq \eta/2$, which proves that $P_{x,\xi} := \text{Id} + V_{x,\xi} + W_{x,\xi}$ is invertible with $\det P_{x,\xi} \geq \frac{1}{2}$. As a consequence its inverse $P_{x,\xi}^{-1} = (\det P_{x,\xi})^{-1} {}^t \text{cof}(P_{x,\xi})$ is also in E_σ^0 . We deduce then that $(\mathbb{R}^n, +, d\lambda, \psi)$ has a S_σ -bounded geometry. With $r(x, \xi, \xi') = -\bar{\psi}(x, \psi(\psi(x, -\xi), -\xi'))$, we get

$$r(x, \xi, \xi') = -\langle x \rangle^\sigma s^{-1}\left(s\left(\frac{-\xi}{\langle x \rangle^\sigma}\right) + \frac{\langle \psi(x, -\xi) \rangle^\sigma}{\langle x \rangle^\sigma} s\left(\frac{-\xi'}{\langle \psi(x, -\xi) \rangle^\sigma}\right)\right).$$

so that $(dr_{x,\xi})_{\xi'} = (ds^{-1} \circ w)(ds \circ u)$ where $w(x, \xi, \xi') := s\left(\frac{-\xi}{\langle x \rangle^\sigma}\right) + v(x, \xi, \xi')$, $v(x, \xi, \xi') := \frac{\langle \psi(x, -\xi) \rangle^\sigma}{\langle x \rangle^\sigma} s\left(\frac{-\xi'}{\langle \psi(x, -\xi) \rangle^\sigma}\right)$, $u(x, \xi, \xi') := -\frac{\xi'}{\langle \psi(x, -\xi) \rangle^\sigma}$. We check that v satisfies

$$\partial^{(\mu, \gamma)} v = \mathcal{O}\left(\langle \psi(x, -\xi) \rangle^{-\sigma|\gamma|} \langle x \rangle^{-\sigma(|\mu|+1)} \langle \zeta \rangle^{\kappa_1|\mu|} \langle \zeta' \rangle^{|\mu|+1}\right).$$

It follows from Peetre's inequality that for any $\varepsilon \in [0, 1]$ and $x, y \in \mathbb{R}^n$, $\langle x + y \rangle \geq 2^{-\varepsilon/2} \frac{\langle x \rangle^\varepsilon}{\langle y \rangle^\varepsilon}$, which implies that $\langle \psi(x, -\xi) \rangle^\sigma = \mathcal{O}(\langle x \rangle^{-\sigma\varepsilon} \langle \xi \rangle^{\sigma\varepsilon})$. As a consequence we get the estimates

$$\begin{aligned} \partial^{(\mu, \gamma)} w &= \mathcal{O}\left(\langle x \rangle^{-\sigma(1+|\mu|+\varepsilon|\gamma|)} \langle \zeta \rangle^{\kappa_1|\mu|+\varepsilon|\gamma|+\delta_{\gamma,0}} \langle \zeta' \rangle^{|\mu|+1}\right), \\ \partial^{(\mu, \gamma)} u &= \mathcal{O}\left(\langle x \rangle^{-\sigma(|\mu|+\varepsilon|\gamma|)} \langle \zeta \rangle^{\kappa_1|\mu|+\varepsilon|\gamma|} \langle \zeta' \rangle^{1-|\gamma|}\right). \end{aligned}$$

We deduce from this that (C_σ) is satisfied. \square

We also check that the hypothesis (H_V) of section 5.4.5 is satisfied so that the previous pseudodifferential calculus (for $\lambda \in \{0, 1\}$) is then valid on $(\mathbb{R}^n, +, d\lambda, \psi)$, and proves in particular the space of operators of the form

$$A(v)(x) = \int_{\mathbb{R}^{2n}} e^{2\pi i \langle \theta, \xi \rangle} a(x, \theta) v(\psi(x, -\xi)) d\xi d\theta = \int_{\mathbb{R}^{2n}} e^{-2\pi i \langle \theta, \bar{\psi}_x(y) \rangle} a(x, \theta) v(y) |J(\bar{\psi}_x)|(y) dy d\theta$$

where $a \in S_\sigma^\infty(\mathbb{R}^{2n})$, is equal to the standard algebra of pseudodifferential operators \mathbb{R}^n . However, since (C_σ) is satisfied, we have now at our disposal a new symbol composition formula given by Theorem 5.4.47, adapted to the new linearization ψ .

5.5.2 S_1 -geometry of the Hyperbolic plane

The (hyperboloid model of the) 2-dimensional hyperbolic space is defined as the submanifold $\mathbb{H} := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 - x_3^2 = -1 \text{ and } x_3 > 0\}$ of the $(2, 1)$ -Minkowski space $\mathbb{R}^{2,1}$ with the bilinear symmetric form $\langle v, w \rangle_{2,1} = v_1 w_1 + v_2 w_2 - v_3 w_3$. The induced metric on \mathbb{H} : $ds^2 = (dx_1)^2 + (dx_2)^2 - (dx_3)^2$ is Riemannian and it is known that \mathbb{H} is a symmetric Cartan–Hadamard manifold with constant negative sectional curvature (equal to -1). The map $\varphi : \mathbb{R}^2 \rightarrow \mathbb{H}$ given by

$$\varphi(x, y) := (\sinh x, \cosh x \sinh y, \cosh x \cosh y)$$

is a diffeomorphism with inverse $\varphi^{-1}(x_1, x_2, x_3) = (\operatorname{argsh} x_1, \operatorname{argsh}(\frac{x_2}{\cosh(\operatorname{argsh} x_1)}))$. As a consequence we can construct another model of the hyperbolic space, noted R^2 with domain \mathbb{R}^2 and metric obtained by pulling back the metric on \mathbb{H} onto \mathbb{R}^2 . A computation shows that this metric is $ds^2 := (dx)^2 + \cosh^2 x (dy)^2$. We will note $\|\cdot\|_p$ the norm on $T_p R^2 \simeq \mathbb{R}^2$ given by this metric, where p is a point in \mathbb{R}^2 , and $\|\cdot\|$ is the Euclidian norm. The geodesic equation on R^2 leads to the following system of ordinary differential equations:

$$\begin{aligned} x'' - \cosh x \sinh x (y')^2 &= 0, \\ y'' + 2 \tanh x x' y' &= 0. \end{aligned} \quad (5.34)$$

For each $p = (x, y) \in \mathbb{R}^2$ and $v \in \mathbb{R}^2$ such that $\|v\|_p = 1$ there exists an unique solution on \mathbb{R} $\gamma_{p,v} = (x(t), y(t))$ of (5.34) such that $\gamma_{p,v}(0) = p$ and $\gamma'_{p,v}(0) = v$.

At each point $p = (x, y) \in \mathbb{R}^2$, we can define the ellipse of unit vectors centered at 0 in $T_p R^2 \simeq \mathbb{R}^2$ with equation $X^2 + (\cosh^2 x) Y^2 = 1$. The polar equation of this ellipse is

$$e_p(\theta) := \frac{1}{\sqrt{1 + \sinh^2 x \sin^2 \theta}}.$$

Thus, any tangent vector $v \in T_p R^2$ with decomposition $v = \|v\| (\cos \theta, \sin \theta)$ also admits the following polar decomposition $v = \|v\|_p (\cos_p \theta, \sin_p \theta)$ where $\cos_p \theta := e_p(\theta) \cos \theta$ and $\sin_p \theta := e_p(\theta) \sin \theta$. Remark that e_p, \cos_p, \sin_p and $\|\cdot\|_p$ are in fact independent of the second coordinate y of p . We shall therefore also use the notations $e_x := e_{(x,y)}$ and similarly for \cos_x, \sin_x and $\|\cdot\|_x$. Note that for any vector $v := \|v\| (\cos \theta, \sin \theta)$, we have $\|v\|_x = \|v\| / e_x(\theta)$.

If $p \in \mathbb{H}$ and $v \in \mathbb{R}^{2,1}$ are such that $\langle p, v \rangle_{2,1} = 0$ and $\langle v, v \rangle_{2,1} = 1$, then the unique geodesic $\alpha_{p,v}$ on \mathbb{H} such that $\alpha_{p,v}(0) = p$ and $\alpha'_{p,v}(0) = v$ is $\alpha_{p,v}(t) = \cosh t p + \sinh t v$ (see for instance [85, p.195]). As a consequence, the geodesics $\gamma_{p,v}$ on the R^2 hyperbolic space can be obtained by pushing forward the $\alpha_{p,v}$ geodesics with the diffeomorphic isometry φ . We check after tedious calculations that for any given $p = (x, y) \in \mathbb{R}^2$ and $\theta \in \mathbb{R}$, the following curve

$$\begin{aligned} \gamma_{p,\theta}^1(t) &= \operatorname{argsh} (\cosh t \sinh x + \sinh t \cosh x \cos_x \theta), \\ \gamma_{p,\theta}^2(t) &= \operatorname{argsh} \left(\frac{\cosh t \cosh x \sinh y + \sinh t (\sinh x \sinh y \cos_x \theta + \cosh x \cosh y \sin_x \theta)}{\cosh (\operatorname{argsh}(\cosh t \sinh x + \sinh t \cosh x \cos_x \theta))} \right), \end{aligned} \quad (5.35)$$

where $t \in \mathbb{R}$, is the unique maximal solution of the geodesic system (5.34) satisfying the initial conditions: $\gamma_{p,\theta}(0) = p$ and $\gamma'_{p,\theta}(0) = (\cos_x(\theta), \sin_x(\theta))$. An explicit formula for the exponential map at any point can therefore be obtained, since we have $\exp_p(v) = \gamma_{p,\theta}(\|v\|_x)$ where $v \in T_p \mathbb{R}^2 - \{0\}$ and $\theta \in \mathbb{R}$ such that $v = \|v\|(\cos \theta, \sin \theta)$. The main interest of this hyperbolic model with domain equal to \mathbb{R}^2 is that it is possible to find explicitly the logarithmic map (the inverse of the exponential map) at any point. We find, after an elementary but long computation, the following inverse, for any $p = (x, y)$ and $p' = (x', y') \in \mathbb{R}^2$,

$$\begin{aligned} \exp_p^{-1}(p') &= \frac{\operatorname{argch} f_p(p')}{\sqrt{(f_p(p'))^2 - 1}} \begin{pmatrix} -g_p(p') \\ \cosh x' \operatorname{sech} x \sinh(y' - y) \end{pmatrix}, \\ f_p(p') &:= \cosh(x') \cosh(y' - y) \cosh(x) - \sinh(x') \sinh(x), \\ g_p(p') &:= \cosh(x') \cosh(y' - y) \sinh(x) - \sinh(x') \cosh(x). \end{aligned} \quad (5.36)$$

We have $\|\exp_p^{-1}(p')\|_p = \operatorname{argch} f_p(p')$ which is the geodesic distance between two arbitrary points p, p' in the \mathbb{R}^2 hyperbolic model. The goal of this section is to prove the following result.

Theorem 5.5.2. \mathbb{H} has a S_1 -bounded geometry.

We note $\mathbb{R}_C^2 := \mathbb{R}^2 \setminus]-\infty, 0] \times \{0\}$ and $\mathbb{R}_P^2 :=]0, +\infty[\times]-\pi, \pi[$. For any $x \in \mathbb{R}$, the map $\chi_x : \mathbb{R}_C^2 \rightarrow \mathbb{R}_P^2$ given by $\chi_x(v_1, v_2) := (\|v\|_x, \arctan(v_1, v_2))$ where $\arctan(v_1, v_2)$ is the unique element θ of $] -\pi, \pi[$ such that $v_1 + iv_2 = \|v\| \exp(i\theta)$, is a diffeomorphism with inverse $\chi_x^{-1}(r, \theta) = (r \cos_x \theta, r \sin_x \theta)$.

Lemma 5.5.3. Let $x \in \mathbb{R}$ and $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$ such that $f \circ \chi_x^{-1} \in C^\infty(\mathbb{R}_P^2, \mathbb{R})$ satisfies for any $(\alpha, \beta) \in \mathbb{N}^2 \setminus \{(0, 0)\}$, and $(r, \theta) \in \mathbb{R}_P^2$, $|\partial^{\alpha, \beta} f \circ \chi_x^{-1}(r, \theta)| \leq C_{\alpha, \beta} \langle r \rangle^{1-\alpha}$ where $C_{\alpha, \beta} > 0$. Then $f \in G_1(\mathbb{R}^2, \mathbb{R})$.

Proof. By Theorem 5.2.11, for any $(\alpha, \beta) \in \mathbb{N}^2 \setminus \{(0, 0)\}$, $\partial^{\alpha, \beta} f = \sum_{1 \leq |\alpha', \beta'| \leq |(\alpha, \beta)|} (\partial^{\alpha', \beta'} f \circ \chi_x^{-1}) \circ \chi_x P_{\alpha, \beta, \alpha', \beta'}(\chi_x)$ on \mathbb{R}_C^2 , where $P_{\alpha, \beta, \alpha', \beta'}(\chi_x)$ is a linear combination of functions of the form $\prod_{j=1}^s (\partial^{l^j} \chi_x)^{k^j}$ where $s \in \{1, \dots, \alpha + \beta\}$. The k^j and l^j are 2-multi-indices (for $1 \leq j \leq s$) such that $|k^j| > 0$, $\sum_{j=1}^s k^j = (\alpha', \beta')$ and $\sum_{j=1}^s |k^j| l^j = (\alpha, \beta)$. By definition, $\chi_x(v) = (\chi_x^1(v), \chi_x^2(v)) = (\|v\|_x, \arctan(v_1, v_2))$. It is straightforward to check that for any 2-multi-index ν , $|\partial^\nu \chi_x^1(v)| \leq C_\nu \langle v \rangle^{1-|\nu|}$ and $|\partial^\nu \chi_x^2(v)| \leq C'_\nu \langle v \rangle^{-|\nu|}$ on \mathbb{R}_C^2 . As a consequence, for each $\alpha, \beta, \alpha', \beta'$ with $1 \leq \alpha' + \beta' \leq \alpha + \beta$ there exists $C_{\alpha, \beta, \alpha', \beta'} > 0$ such that for any $v \in \mathbb{R}_C^2$,

$$|P_{\alpha, \beta, \alpha', \beta'}(\chi_x)(v)| \leq C_{\alpha, \beta, \alpha', \beta'} \langle v \rangle^{\alpha' - (\alpha + \beta)}.$$

Moreover, by hypothesis, there is $C_{\alpha', \beta'} > 0$ such that for any $v \in \mathbb{R}_C^2$, $|(\partial^{\alpha', \beta'} f \circ \chi_x^{-1}) \circ \chi_x(v)| \leq C_{\alpha', \beta'} \langle v \rangle^{1-\alpha'}$. This gives $f \in G_1(\mathbb{R}_C^2, \mathbb{R})$. The extension to $G_1(\mathbb{R}^2, \mathbb{R})$ is a direct consequence of the smoothness of f on \mathbb{R}^2 and the fact that \mathbb{R}_C^2 is dense in \mathbb{R}^2 . \square

We shall use the following proposition, which gives a formal expression of the successive derivatives of the inverse (and its real powers) of a smooth function.

Proposition 5.5.4. Let $s > 0$ be given. For any nonzero n -multi-index ($n \in \mathbb{N}^*$) α , there exist a finite nonempty set J_α , nonzero real numbers $(\lambda_{s, \alpha, p})_{p \in J_\alpha}$ and n -multi-indices $\beta^{\alpha, p, j}$ (with $p \in J_\alpha$, $1 \leq j \leq |\alpha|$) such that
- for any $p \in J_\alpha$, $\sum_{1 \leq j \leq |\alpha|} \beta^{\alpha, p, j} = \alpha$,

- for any smooth function $f \in C^\infty(\mathbb{R}^n, \mathbb{R}_+^*)$,

$$\partial^\alpha \frac{1}{f^s} = \frac{1}{f^{|\alpha|+s}} \sum_{p \in J_\alpha} \lambda_{s,\alpha,p} \prod_{j=1}^{|\alpha|} \partial^{\beta^{\alpha,p,j}} f.$$

Proof. The result is true for the case $|\alpha| = 1$. Suppose then that the result holds for any n -multi-index α such that $|\alpha| = k$, where $k \in \mathbb{N}^*$ and let α' be a n -multi-index such that $|\alpha'| = k + 1$. Let i be the smallest element of $\{1, \dots, n\}$ such that $\alpha'_i \geq 1$, and set $\alpha := (\alpha'_1, \dots, \alpha'_{i-1}, \alpha'_i - 1, \alpha'_{i+1}, \dots, \alpha'_n)$. Thus for any $f \in C^\infty(\mathbb{R}^n, \mathbb{R}_+^*)$, $\partial^{\alpha'} \frac{1}{f^s} = \partial_i \partial^\alpha \frac{1}{f^s}$. Since $|\alpha| = k$, there exist a finite nonempty set J_α , nonzero real numbers $(\lambda_{s,\alpha,p})_{p \in J_\alpha}$ and n -multi-indices $\beta^{\alpha,p,j}$ (with $p \in J_\alpha$, $1 \leq j \leq |\alpha|$) such that for any $p \in J_\alpha$, $\sum_{1 \leq j \leq |\alpha|} \beta^{\alpha,p,j} = \alpha$, and such that for any $f \in C^\infty(\mathbb{R}^n, \mathbb{R}_+^*)$, $\partial^\alpha \frac{1}{f^s} = \frac{1}{f^{|\alpha|+s}} \sum_{p \in J_\alpha} \lambda_{s,\alpha,p} \prod_{j=1}^{|\alpha|} \partial^{\beta^{\alpha,p,j}} f$. As a consequence, with the formula $\partial_i \prod_{j=1}^{|\alpha|} g_j = \sum_{q=1}^{|\alpha|} \prod_{j=1}^{|\alpha|} \partial^{\delta_{q,j} e_i} g_j$, we obtain for any $f \in C^\infty(\mathbb{R}^n, \mathbb{R}_+^*)$,

$$\partial^{\alpha'} \frac{1}{f^s} = \frac{1}{f^{|\alpha'|+s}} \left(\sum_{p \in J_\alpha} -(|\alpha| + s) \lambda_{s,\alpha,p} \left(\prod_{j=1}^{|\alpha|} \partial^{\beta^{\alpha,p,j}} f \right) \partial_i f + \sum_{(p,q) \in J_\alpha \times \mathbb{N}_{|\alpha|}} \lambda_{s,\alpha,p} \left(\prod_{j=1}^{|\alpha|} \partial^{\delta_{q,j} e_i + \beta^{\alpha,p,j}} f \right) f \right).$$

Thus, if we take $J_{\alpha'} = J_\alpha \amalg (J_\alpha \times \mathbb{N}_{|\alpha|})$, $\lambda_{s,\alpha',\tilde{p}} := -(s + |\alpha|) \lambda_{s,\alpha,p}$ if $\tilde{p} = p \in J_\alpha$, $\lambda_{s,\alpha',\tilde{p}} := \lambda_{s,\alpha,p}$ if $\tilde{p} = (p, q) \in J_\alpha \times \mathbb{N}_{|\alpha|}$, $\beta^{\alpha',\tilde{p},j} := \beta^{\alpha,p,j}$ if $\tilde{p} = p \in J_\alpha$ and $1 \leq j \leq |\alpha|$, $\beta^{\alpha',\tilde{p},j} := e_i$ if $\tilde{p} = p \in J_\alpha$ and $j = |\alpha| + 1 = |\alpha'|$, $\beta^{\alpha',\tilde{p},j} := \delta_{q,j} e_i + \beta^{\alpha,p,j}$ if $\tilde{p} = (p, q) \in J_\alpha \times \mathbb{N}_{|\alpha|}$ and $1 \leq j \leq |\alpha|$ and $\beta^{\alpha',\tilde{p},j} := 0$ if $\tilde{p} = (p, q) \in J_\alpha \times \mathbb{N}_{|\alpha|}$ and $j = |\alpha| + 1 = |\alpha'|$, the result now holds for α' . \square

In the following we set the convention $J_0 := \{1\}$, $\lambda_{s,0,1} := 1$ and $\prod_{j=1}^0 := 1$, so that the formula giving $\partial^\alpha \frac{1}{f^s}$ in the previous lemma is still valid when $\alpha = 0$. When $s \in \mathbb{N}^*$, the result is also valid for complex valued nowhere zero smooth functions.

We note H_P the space of $C^\infty(\mathbb{R}_P^2, \mathbb{R})$ functions of the form $(r, \theta) \mapsto a(\theta) \cosh r + b(\theta) \sinh r$ where $a, b \in \mathcal{B}(\mathbb{R})$, and $A_{P,k}$ the space of functions $f \in C^\infty(\mathbb{R}_P^2, \mathbb{R})$ such that for any 2-multi-index (α, β) with $\alpha \leq k \in \mathbb{N}$, there is $C_{\alpha,\beta} > 0$ such that for any $(r, \theta) \in \mathbb{R}_P^2$, $|\partial^{\alpha,\beta} f(r, \theta)| \leq C_{\alpha,\beta} \langle r \rangle^{k-\alpha}$, and also such that for any 2-multi-index (α, β) with $\alpha \geq k+1$, there is $C'_{\alpha,\beta} > 0$ such that for any $(r, \theta) \in \mathbb{R}_P^2$, $|\partial^{\alpha,\beta} f(r, \theta)| \leq C'_{\alpha,\beta} e^{-2r}$. Clearly, $A_{P,k} \subset S_{P,k}$ where $S_{P,k}$ is the space of functions $f \in C^\infty(\mathbb{R}_P^2, \mathbb{R})$ such that for any 2-multi-index (α, β) , there is $C_{\alpha,\beta} > 0$ such that for any $(r, \theta) \in \mathbb{R}_P^2$, $|\partial^{\alpha,\beta} f(r, \theta)| \leq C_{\alpha,\beta} \langle r \rangle^{k-\alpha}$. By Leibniz rule, $S_{P,k} S_{P,k'} \subseteq S_{P,k+k'}$. We note N_P the space of functions $f \in C^\infty(\mathbb{R}_P^2, \mathbb{R})$ such that for any 2-multi-index (α, β) there is $C_{\alpha,\beta} > 0$ such that for any $(r, \theta) \in \mathbb{R}_P^2$, $|\partial^{\alpha,\beta} f(r, \theta)| \leq C_{\alpha,\beta} e^{-2r}$. If $r_0 > 0$ we define the spaces H_{P,r_0} , A_{P,k,r_0} , S_{P,k,r_0} and N_{P,r_0} exactly as before, except that we now replace the domain \mathbb{R}_P^2 by $\mathbb{R}_{P,r_0}^2 :=]r_0, +\infty[\times]-\pi, \pi[$.

Lemma 5.5.5. *Let $f, g, h, w \in H_{P,r_0}$ where $r_0 > 0$, such that there is $\varepsilon > 0$, $C > 1$ such that for any $(r, \theta) \in \mathbb{R}_{P,r_0}^2$, $f \geq C$, $f \geq \varepsilon e^r$ and $h^2 + g^2 \geq \varepsilon e^{2r}$.*

- (i) *The functions $\frac{w}{(h^2+g^2)^{3/2}}$, $\frac{w}{(f^2-1)^{3/2}}$ and any function of the form $(r, \theta) \mapsto \frac{\sum_{k=-4}^4 b_k(\theta) e^{kr}}{((h^2+g^2)(1+h^2+g^2))^{3/2}}$, where $b_k \in \mathcal{B}(\mathbb{R})$, are in N_{P,r_0} .*
- (ii) *The functions $\argch \sqrt{1+h^2+g^2}$ and $\argch f$ are in $A_{P,1,r_0}$.*
- (iii) *The functions $\frac{w}{\sqrt{h^2+g^2}}$ and $\frac{w}{\sqrt{f^2-1}}$ are in $A_{P,0,r_0}$.*

Proof. (i) We give a proof for $\frac{w}{(h^2+g^2)^{3/2}}$. The other cases are similar. By Proposition 5.5.4 and Leibniz rule, we have for any 2-multi-index ν ,

$$\partial^\nu \frac{w}{(h^2+g^2)^{3/2}} = \sum_{\nu' \leq \nu} \binom{\nu}{\nu'} \frac{\partial^{\nu-\nu'} w}{(h^2+g^2)^{3/2+|\nu'|}} \sum_{p \in J_{\nu'}} \lambda_{3/2, \nu', p} \prod_{j=1}^{|\nu'|} \partial^{\beta^{\nu', p, j}} (h^2 + g^2).$$

Note that we have for any 2-multi-index ν , $\partial^\nu (h^2 + g^2) = \mathcal{O}(e^{2r})$ and $\partial^\nu w = \mathcal{O}(e^r)$. The result follows.

(ii) By (i), since $\partial_r^2 \operatorname{argch} \sqrt{1+h^2+g^2}$ is of the form $(r, \theta) \mapsto \frac{\sum_{k=-4}^4 b_k(\theta) e^{kr}}{((h^2+g^2)(1+h^2+g^2))^{3/2}}$ where $b_k \in \mathcal{B}(\mathbb{R})$, and $\partial_r^2 \operatorname{argch} f$ is of the form $\frac{w}{(f^2-1)^{3/2}}$ where $w \in H_{P, r_0}$, we only need to check that for $0 \leq \alpha \leq 1$, and $\beta \in \mathbb{N}$, $\partial^{\alpha, \beta} \operatorname{argch} \sqrt{1+h^2+g^2} = \mathcal{O}(\langle r \rangle^{1-\alpha})$ and $\partial^{\alpha, \beta} \operatorname{argch} f = \mathcal{O}(\langle r \rangle^{1-\alpha})$. Since $\partial_r \operatorname{argch} \sqrt{1+h^2+g^2} = \frac{(\partial_r h)h + (\partial_r g)g}{\sqrt{(h^2+g^2)(1+h^2+g^2)}}$, $\partial_r \operatorname{argch} f = \frac{\partial_r f}{\sqrt{f^2-1}}$, $\partial_\theta \operatorname{argch} \sqrt{1+h^2+g^2} = \frac{(\partial_\theta h)h + (\partial_\theta g)g}{\sqrt{(h^2+g^2)(1+h^2+g^2)}}$ and $\partial_\theta \operatorname{argch} f = \frac{\partial_\theta f}{\sqrt{f^2-1}}$, the result follows from an application of Proposition 5.5.4.

(iii) By (i), since $\partial_r \frac{w}{\sqrt{h^2+g^2}}$ is of the form $\frac{w_1}{(h^2+g^2)^{3/2}}$ where $w_1 \in H_{P, r_0}$, and $\partial_r \frac{w}{\sqrt{f^2-1}}$ is of the form $\frac{w_2}{(f^2-1)^{3/2}}$ where $w_2 \in H_{P, r_0}$, we only need to check that for $\beta \in \mathbb{N}$, $\partial^{0, \beta} \frac{w}{\sqrt{h^2+g^2}} = \mathcal{O}(1)$ and $\partial^{0, \beta} \frac{w}{\sqrt{f^2-1}} = \mathcal{O}(1)$. This is a direct consequence of Proposition 5.5.4. \square

Proof of Theorem 5.5.2. By Lemma 5.2.14 (iii) and Proposition 5.2.12, it is sufficient to prove that for any $p := (x, y) \in \mathbb{R}^2 \setminus \{0\}$, $\exp_p^{-1} \circ \exp_0$ and $\exp_0^{-1} \circ \exp_p$ are in $G_1(\mathbb{R}^2)$. A computation based on (5.35) and (5.36) shows that on \mathbb{R}_P^2 ,

$$\begin{aligned} \exp_p^{-1} \circ \exp_0 \circ \chi_0^{-1} &= (\operatorname{argch} f) \left(\frac{w_1}{\sqrt{f^2-1}}, \frac{w_2}{\sqrt{f^2-1}} \right), \\ \exp_0^{-1} \circ \exp_p \circ \chi_x^{-1} &= (\operatorname{argch} \sqrt{1+h^2+g^2}) \left(\frac{h}{\sqrt{h^2+g^2}}, \frac{g}{\sqrt{h^2+g^2}} \right), \end{aligned}$$

where

$$\begin{aligned} f(r, \theta) &:= \cosh r \cosh y \cosh x - \sinh r (\sinh x \cos \theta + \sinh y \cosh x \sin \theta), \\ w_1(r, \theta) &:= -\cosh r \cosh y \sinh x + \sinh r (\cosh x \cos \theta + \sinh y \sinh x \sin \theta), \\ w_2(r, \theta) &:= -\cosh r \sinh y \operatorname{sech} x + \sinh r \sin \theta \cosh y \operatorname{sech} x, \\ h(r, \theta) &:= \cosh r \sinh x + \sinh r \cosh x \cos_x \theta, \\ g(r, \theta) &:= \cosh r \cosh x \sinh y + \sinh r (\sinh x \sinh y \cos_x \theta + \cosh x \cosh y \sin_x \theta). \end{aligned}$$

All these functions belong to H_P and $f \geq 1$. Note that $f(r, \theta) = 1$ if and only if $\exp_0 \chi_0^{-1}(r, \theta) = p$, in which case $\exp_p^{-1} \circ \exp_0 \circ \chi_0^{-1}(r, \theta) = 0$, so that $\exp_p^{-1} \circ \exp_0 \circ \chi_0^{-1}$ is well defined as a smooth function on the whole \mathbb{R}_P^2 . The same argument holds for $\exp_0^{-1} \circ \exp_p \circ \chi_x^{-1}$. We check that

$$\frac{1}{2}(\cosh x \cosh y - \sqrt{\cosh^2 x \cosh^2 y - 1})e^r \leq f(r, \theta) \leq \cosh r e^{\operatorname{argch}(\cosh x \cosh y)}$$

so that by defining $r_0 := \log 2/\varepsilon$ where $0 < \varepsilon < \min\{1, \frac{1}{2}(\cosh x \cosh y - \sqrt{\cosh^2 x \cosh^2 y - 1})\}$ we have for any $(r, \theta) \in \mathbb{R}_{P, r_0}^2$, $f(r, \theta) \geq \varepsilon e^r \geq 2$. Note also that for any $v \in \mathbb{R}_C^2$, we have

$\argch f(\chi_0(v)) = \|\exp_p^{-1} \circ \exp_0(v)\|_p$ and

$$\argch \sqrt{1 + h^2(\chi_x(v)) + g^2(\chi_x(v))} = \|\exp_0^{-1} \circ \exp_p(v)\|_0 .$$

The first equality entails (since $\exp_p^{-1} \circ \exp_0(\mathbb{R}_C^2)$ is a dense open subset of \mathbb{R}^2) that for any v in \mathbb{R}^2 , $\cosh \|v\|_p \leq \cosh \|\exp_0^{-1} \circ \exp_p(v)\|_0 e^{\argch(\cosh x \cosh y)}$. We then obtain for any $(r, \theta) \in \mathbb{R}_P^2$, $\sqrt{1 + h^2 + g^2} \geq \cosh r e^{-\argch(\cosh x \cosh y)}$. In particular, defining

$$r'_0 := \argch(\sqrt{2} \exp(\argch(\cosh x \cosh y))),$$

we get for any $r \geq r'_0$, the following estimate $h^2 + g^2 \geq \frac{1}{8} e^{-2 \argch(\cosh x \cosh y)} e^{2r}$. If we now apply Lemma 5.5.5 for the space H_{P, r'_0} where $r''_0 := \max\{r_0, r'_0\}$, we see that $\exp_p^{-1} \circ \exp_0 \circ \chi_0^{-1}$ and $\exp_0^{-1} \circ \exp_p \circ \chi_x^{-1}$ are in $S_{P,1}$. The result then follows from Lemma 5.5.3. \square

Conclusion

The Chamseddine–Connes spectral action is a fundamental object that lies at the interface of noncommutative geometry and its applications in particle physics. In this thesis, we studied the spectral action in noncommutative spaces such as the noncommutative torus and the quantum group $SU_q(2)$. We have also been interested in some mathematical questions about commutative geometries (compact spin manifolds) and deformation quantization on manifolds with linearization. In all these works, pseudodifferential calculus (on abstract spectral triples or on manifolds) has played a crucial role. It is with the help of the fundamental notion of pseudodifferential operators that the spectral actions on the noncommutative torus and on $SU_q(2)$ have been computed.

In chapter 2, we presented a computation of the spectral action on the n -noncommutative torus $(C^\infty(\mathcal{T}_\Theta^n), \mathcal{H}, \mathcal{D})$, which is a simple spectral triple, even when pseudodifferential operators take into account the JAJ^{-1} operators. The spectral action in dimension 4 shows that a new noncommutative Yang–Mills term appears: $\tau(F_{\mu\nu}F^{\mu\nu})$, where $F_{\mu\nu} := \delta_\mu(A_\nu) - \delta_\nu(A_\mu) - [A_\mu, A_\nu]$, which extends the usual commutative $F_{\mu\nu} := \delta_\mu(A_\nu) - \delta_\nu(A_\mu)$. A number theoretical condition on the deformation matrix Θ , related to Diophantine approximation theory, is crucial in the computation of the spectral action, when the perturbation $D \rightarrow D + A + \varepsilon JAJ^{-1}$ is considered. An interesting question remains: is this Diophantine condition really necessary to obtain this spectral action, with exactly the same Yang–Mills term $\tau(F_{\mu\nu}F^{\mu\nu})$? We conjecture that it is, as suggested by some heuristic arguments (Remark 2.4.7).

We presented in chapter 3 the computation of the spectral action on the spectral triple of [48] associated to the quantum group $SU_q(2)$. Once again, we took into account the real structure J and used the pseudodifferential techniques previously described in sections 1.2 and 1.3. However, contrarily to the case of the noncommutative torus, some remarkable new phenomena appear in this q -deformed noncommutative space. First, the dimension spectrum of $SU_q(2)$ is bounded below, which implies that there is only a finite number of terms in the spectral action expansion. Moreover, in this space, tadpoles do exist whereas they vanish on the noncommutative torus. We also found that the limit $q \rightarrow 1$ (which corresponds to a limit from the quantum 3-sphere towards the commutative 3-sphere) of the spectral action does not exist automatically, and when it exists, such limit does not lead to the associated action on the commutative sphere \mathbb{S}^3 . All these facts show that there is a “wall” between q -deformed geometries and the commutative world, that is nonexistent in the Θ -deformation of tori or Moyal planes. Naturally, it would be interesting to investigate other related cases like the quantum projective plane [45], Podleś spheres [43, 46] or the Euclidean quantum spheres [47, 96], especially the 4-sphere [44].

In the fourth chapter, we investigated possible cancellations of terms in the Chamseddine–Connes spectral action formula in commutative Riemannian spectral triples. We showed in particular that there are no tadpoles, i.e. terms of the form $\oint A\mathcal{D}^{-1}$ are zero. More generally, the tadpoles are the A -linear terms in the internal fluctuation of the spectral action $S(\mathcal{D}_A, \Phi, \Lambda)$. We considered, after Chamseddine and Connes [23], the case of a chiral boundary condition on the Dirac operator in a spin manifold with boundary. It turns out that, in this case, there are no tadpoles up to order 5. However, this approach is based on explicit computations of first heat kernel coefficients associated to a mixed boundary condition. We expect that there are no tadpoles at any order for a chiral boundary condition and shall investigate this question, using a spectral triple approach, in a subsequent paper [84].

We have seen, in chapter 5, certain hypotheses on the geometry (S_σ or \mathcal{O}_M -bounded geometry) of a manifold with linearization that allow a coordinate free definition of most of the topological vector spaces needed for Fourier analysis and global complete symbol calculus with uniform and decaying control over the x variable. Given a linearization on the manifold with some properties of control at infinity, we constructed symbol maps and λ -quantization, explicit Moyal star-products on the cotangent bundle, and classes of pseudodifferential operators. We proved a result of stability under composition, and an associated symbol product asymptotic formula under a hypothesis (C_σ) of control at infinity of the linearization. The calculus presented here is a generalization of the standard and SG symbol calculi over the Euclidean space \mathbb{R}^n and can be applied to the hyperbolic 2-space since, as proven in section 5.2, it has a S_1 -bounded geometry. L^2 -continuity of pseudodifferential operators of order $(0, 0)$ has been established in section 5.4.5 under the hypothesis (H_V). We do not know however if this result still holds without this hypothesis. The full analysis of the obtained Moyal product on $\mathcal{S}(T^*M)$ and spectral properties of pseudodifferential operators in $\Psi_\sigma^{l,m}$ remain to be studied. The main goal would be the construction of noncommutative noncompact spectral triples based on the algebra $(\mathcal{S}(T^*M), \circ_W)$, which could extend the spectral triple described in [58]. Moreover, extension and connection of the symbol calculus presented here could be made with Fourier integral operators [42, 117, 118], regularized traces [110] and Gelfand–Shilov spaces [15].

Finally, the spectral actions that we computed are classical, and quantization through functional integration of these actions still remains an open and challenging problem.

Résumé de la thèse en français

Cette thèse constitue un recueil des travaux de recherche que j'ai effectués ces trois dernières années en collaboration avec Driss Essouabri, Bruno Iochum et Andrzej Sitarz.

La géométrie non commutative est un vaste domaine des mathématiques dont l'objet est la généralisation de l'ensemble des concepts apparaissant en géométrie classique. Plus particulièrement, à l'aide d'un formalisme issu de l'analyse fonctionnelle, de la théorie des algèbres d'opérateurs, de la théorie spectrale et de la géométrie spinorielle, la géométrie non commutative généralise notamment les concepts d'espace topologique localement compact, de variété riemannienne orientée compacte à spin et les calculs différentiels et intégraux de la géométrie différentielle. Au-delà de l'intérêt purement mathématique de la géométrie non commutative, il existe des motivations physiques profondes qui poussent les physiciens théoriciens à utiliser ces concepts pour décrire les éléments fondamentaux de la physique (l'espace-temps et les champs). Plus spécifiquement, la géométrie non commutative apparaît comme un cadre mathématique particulièrement adapté à la formulation des concepts quantiques et des processus de quantification.

Il est possible de considérer que la géométrie non commutative (ou tout au moins sa composante topologique) est née lorsqu'a été établi le théorème suivant (premier théorème de Gelfand–Naimark) : toute C^* -algèbre commutative unifère est isométriquement isomorphe à la C^* -algèbre des fonctions continues sur un compact, à savoir l'espace des caractères sur l'algèbre. Étant donné que toute l'information topologique d'un espace est contenue dans l'ensemble des fonctions continues sur cet espace, on peut constater que la notion de C^* -algèbre unifère permet de généraliser la notion d'espace topologique compact. La généralisation non commutative nous fait donc changer de point de vue : ce n'est plus l'ensemble des points (l'espace topologique) qui est fondamental, mais plutôt l'ensemble des fonctions sur l'ensemble des points.

Ce théorème de Gelfand–Naimark a été le point de départ de la géométrie non commutative. À partir de ce résultat fondamental, il a été possible d'étendre la généralisation au-delà des concepts purement topologiques et de construire une véritable géométrie riemannienne non commutative avec ses propres versions non commutatives des notions classiques de calcul différentiel, de calcul intégral, de fibré vectoriel, de mesure, de variétés riemanniennes à spin, etc. Ce travail colossal a été réalisé principalement par Alain Connes [26–30].

Les deux aspects de la géométrie non commutative (non-commutativité et "perte" de la notion de point) ne sont pas sans rappeler la structure fondamentale de la physique quantique. En effet, en physique quantique, les observables ne commutent pas forcément et l'évaluation $f(x)$ d'une observable f en un point x , n'est pas définie. En revanche, la notion d'observable existe

toujours et $x(f)$ a un sens, pourvu que x soit un état, c'est à dire l'équivalent non commutatif du caractère (du point) pouvant évaluer les observables f .

Cette ressemblance frappante entre la géométrie non commutative et la structure de la physique quantique n'est pas anodine et constitue même la source principale qui a motivé le développement de la géométrie non commutative et son application en physique. En particulier, la géométrie non commutative fournit les concepts permettant d'appliquer l'idée fondamentale de la non-commutativité des observables à l'espace-temps lui-même, donnant par là même une de ses motivations fondamentales à la physique.

Les deux théories des interactions fondamentales, à savoir la théorie quantique des champs (modèle standard) pour les interactions électrofaibles et fortes, et la relativité générale pour l'interaction gravitationnelle, n'utilisent pas le même formalisme (la première est quantique, la seconde est classique) et ne voient pas l'espace-temps de la même façon (celui-ci est fixe et minkowskien pour la première, et dynamique pour la seconde). Ces différences fondamentales ne sont pas gênantes pour l'étude des phénomènes favorisant l'interaction gravitationnelle devant les autres ou réciproquement, car ces théories ont chacune un grand succès expérimental dans leur domaine d'application. Cependant, pour l'étude des phénomènes mettant manifestement en jeu toutes les interactions (objets compacts, trous noirs, big-bang), il est nécessaire de rendre compatible ces deux théories, et de les réunir sous un même formalisme. L'idée généralement poursuivie par les théoriciens consiste à généraliser le formalisme de la théorie quantique des champs à la gravitation, autrement dit, réaliser une gravitation quantique. La poursuite de cet objectif s'est développée à travers plusieurs approches différentes, dont notamment la théorie des cordes, qui utilise une augmentation du nombre de dimensions, dont certaines sont compactifiées, et la théorie de la gravité quantique à boucle, qui utilise une structure en "spin foam" pour l'espace-temps sans utiliser de métrique spatio-temporelle en "background" comme le fait la théorie des cordes. Aucune de ces théories n'a reçu de confirmation expérimentale, les prédictions théoriques étant elles-mêmes difficiles.

L'approche suggérée par la géométrie non commutative consiste à utiliser une généralisation non commutative de la variété lorentzienne modélisant l'espace-temps. En introduisant la non-commutativité au niveau même de la structure de l'espace-temps, cette approche permet d'appréhender l'impossibilité de la continuité de l'espace-temps suggérée par la mécanique quantique et la limite intrinsèque que constitue la longueur de Planck $l_p = \sqrt{\frac{\hbar G}{c^3}} \approx 10^{-33}$ cm. Cette approche a notamment permis d'unifier, au niveau classique, les trois interactions du modèle standard avec la gravitation, et d'interpréter géométriquement le mécanisme de Higgs en physique des particules. L'objet central dans l'interface entre la GNC et la physique fondamentale est celui de triplet spectral, généralisation non commutative de la notion de variété riemannienne à spin, point de départ naturel pour l'élaboration de théories physiques. En considérant un triplet spectral dit "presque commutatif", c'est-à-dire un produit d'un triplet spectral commutatif (variété riemannienne compacte, modélisant l'espace-temps "continu") avec un triplet spectral de dimension nulle (une algèbre matricielle), on peut, à l'aide d'une fonctionnelle d'action particulière sur le produit obtenu, appelée action spectrale de Chamseddine–Connes, retrouver le modèle standard et la relativité générale. Plus précisément, l'action spectrale $S = \text{Tr } \Phi(\mathcal{D}/\Lambda)$ permet d'unifier au niveau classique les interactions électro-faible, forte et gravitationnelle [17, 21–25, 33, 37, 86, 87, 131, 136]. Elle est définie à partir du spectre d'un opérateur de Dirac \mathcal{D} et correspond au nombre des valeurs propres de l'opérateur de Dirac inférieures ou égales à une certaine échelle de masse Λ . En procéd-

dant à une fluctuation de la métrique, c'est-à-dire, au niveau algébrique, à une transformation de jauge généralisée au groupe des unitaires de l'algèbre du triplet spectral, il est possible d'obtenir le Lagrangien du modèle standard couplé au Lagrangien gravitationnel d'Einstein–Hilbert, en développant cette fonctionnelle d'action en puissances de Λ .

Dans cette thèse, nous nous sommes intéressés à certaines questions mathématiques associées au calcul d'action spectrale sur certains espaces non commutatifs tels que le tore non commutatif et la 3-sphère non commutative $SU_q(2)$. Nous nous sommes aussi intéressés à l'existence de tadpoles (termes linéaires associés au potentiel A de la fluctuation métrique dans l'action spectrale) dans le cas de géométries riemanniennes commutatives et à la construction d'un calcul symbolique global générant un produit de Wey–Moyal sur les sections rapidement décroissantes d'un fibré d'une variété avec linéarisation. Dans tous ces travaux, l'outil fondamental a été le calcul pseudodifférentiel, qu'il soit abstrait (sur un triplet spectral quelconque), ou symbolique (sur les variétés).

Cette thèse est divisée en cinq parties. Voici un résumé de chacune de ces parties :

1. Action spectrale sur triplets spectraux

Ce chapitre, ainsi que le chapitre suivant, a fait l'objet de la publication *Spectral action on noncommutative torus* [53], qui est un travail en collaboration avec Driss Essouabri, Bruno Iochum et Andrzej Sitarz.

Un triplet spectral est la donnée d'une algèbre involutive \mathcal{A} représentée fidèlement par des opérateurs bornés sur un espace de Hilbert \mathcal{H} et d'un opérateur autoadjoint \mathcal{D} sur \mathcal{H} à résolvante compacte. On demande d'autre part que les commutateurs de $[\mathcal{D}, \mathcal{A}]$ soient bornés. Afin de pouvoir construire un calcul pseudodifférentiel et une théorie de champ non commutative, il est utile d'introduire des hypothèses supplémentaires sur le triplet $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. On dit notamment que le triplet est de dimension n si les valeurs singulières $(\lambda_j)_j$ de l'opérateur \mathcal{D} sont du type $\lambda_j = \mathcal{O}(j^{-1/n})$ et qu'il est régulier si \mathcal{A} et $[\mathcal{D}, \mathcal{A}]$ sont dans $\cap_k \text{Dom } \delta^k$, où $\delta(T) := [|\mathcal{D}|, T]$ (c'est à dire, qu'il est possible de "dériver" tout élément de l'algèbre). Afin d'avoir une théorie contenant un opérateur de conjugaison de charge, il est nécessaire d'introduire une notion de structure réelle sur le triplet spectral: il s'agit d'un opérateur anti-unitaire J qui commute ou anticommute avec \mathcal{D} , selon la dimension du triplet: $\mathcal{D}J = \varepsilon J\mathcal{D}$, avec $\varepsilon \in \{0, 1\}$. Dans cet environnement, les bosons de jauge sont vus comme des fluctuations internes de l'opérateur de Dirac (c'est-à-dire, au niveau classique, de la métrique): $\mathcal{D} \rightarrow \mathcal{D}_A := \mathcal{D} + A + \varepsilon JAJ^{-1}$, où ici A est une 1-forme autoadjointe, c'est à dire une combinaison linéaire d'opérateurs du type $a[\mathcal{D}, b]$, où a et b sont des éléments de l'algèbre \mathcal{A} .

Étant donné un triplet régulier $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ avec structure réelle J , un point fondamental pour faire le lien avec la physique, est d'obtenir une fonctionnelle d'action. Le principe de l'action spectrale de A. Chamseddine et A. Connes dit que la fonctionnelle suivante

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) := \text{Tr}(\Phi(\mathcal{D}_A/\Lambda))$$

où Φ est une fonction de cut-off et Λ un paramètre d'échelle de masse, est la fonctionnelle fondamentale qui peut être utilisée à la fois au niveau classique pour comparer différents espaces géométriques et au niveau quantique dans la formulation par intégrale fonctionnelle, après rotation de Wick à partir de la signature euclidienne.

Afin de pouvoir calculer précisément cette fonctionnelle en fonction de A et en retirer le maximum d'information, il apparait fondamental de pouvoir développer un calcul pseudodifférentiel abstrait sur le triplet $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.

En posant $D := \mathcal{D} + P_0$, où P_0 est la projection orthogonale sur $\text{Ker } \mathcal{D}$, et

$$\begin{aligned} OP^0 &:= \{T : t \mapsto F_t(T) \in C^\infty(\mathbb{R}, \mathcal{B}(\mathcal{H}))\}, \\ OP^\alpha &:= \{T : T|D|^{-\alpha} \in OP^0\}, \end{aligned}$$

où $F_t(T) := e^{it|\mathcal{D}|} T e^{-it|\mathcal{D}|}$, on peut introduire la définition suivante des opérateurs pseudodifférentiels, qui forment une algèbre \mathbb{Z} -graduée, en tenant compte à la fois de la valeur absolue de l'opérateur de Dirac, et de la structure réelle J :

Définition. Soit $\mathcal{D}(\mathcal{A})$ l'algèbre générée par \mathcal{A} , JAJ^{-1} , \mathcal{D} et $|\mathcal{D}|$. Un opérateur pseudodifférentiel est un opérateur T tel qu'il existe $d \in \mathbb{Z}$ tel que pour tout $N \in \mathbb{N}$, il existe $p \in \mathbb{N}_0$, $P \in \mathcal{D}(\mathcal{A})$ et $R \in OP^{-N}$ tels que $P D^{-2p} \in OP^d$ et

$$T = P D^{-2p} + R.$$

Il se trouve que si le triplet spectral est *simple*, c'est dire si les fonctions $s \mapsto \zeta_D^P(s) := \text{Tr}(P|D|^{-s})$, où P est un opérateur pseudodifférentiel d'ordre zéro (la structure réelle J étant prise en compte), sont méromorphes sur \mathbb{C} avec uniquement des pôles simples, alors la fonctionnelle suivante (appelée intégrale non commutative)

$$\oint P := \text{Res}_{s=0} \text{Tr } P|D|^{-s}$$

est une trace sur l'algèbre des opérateurs pseudodifférentiels. Etant donné qu'un développement du type "noyau de la chaleur" (voir par exemple [37, Theorem 1.145]) implique

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = \sum_{0 < k \in Sd^+} \Phi_k \Lambda^k \oint |D_A|^{-k} + \Phi(0) \zeta_{D_A}(0) + \mathcal{O}(\Lambda^{-1}),$$

où $\Phi_k = \frac{1}{2} \int_0^\infty \Phi(t) t^{k/2-1} dt$ et Sd^+ est la partie strictement positive du spectre de dimension de $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ (ensemble des pôles des fonctions ζ_D^P), la principale difficulté est le calcul des termes $\oint |D_A|^{-k}$, $\zeta_{D_A}(0)$. En utilisant le calcul pseudodifférentiel précédent et le fait que l'intégrale non commutative est nulle sur l'espace des opérateurs dans $\mathcal{L}^1(\mathcal{H})$, on peut alors établir les résultats suivants, en notant $\tilde{A} := A + \varepsilon JAJ^{-1}$, $X := \{\tilde{A}, \mathcal{D}\} + \tilde{A}^2$, $\nabla(T) := [\mathcal{D}^2, T]$, $\varepsilon(T) := \nabla(T)D^{-2}$, $g(s, r) := \binom{s/2}{r}$:

$$\begin{aligned} \zeta_{D_A}(0) - \zeta_D(0) &= \sum_{q=1}^n \frac{(-1)^q}{q} \oint (\tilde{A}D^{-1})^q, \\ \oint |D_A|^{-(n-k)} &= \oint |D|^{-(n-k)} + \sum_{p=1}^k \sum_{r_1, \dots, r_p=0}^{k-p} \text{Res}_{s=n-k} h(s, r, p) \text{Tr}(\varepsilon^{r_1}(Y) \cdots \varepsilon^{r_p}(Y) |D|^{-s}), \\ h(s, r, p) &:= (-s/2)^p \int_{0 \leq t_1 \leq \dots \leq t_p \leq 1} g(-st_1, r_1) \cdots g(-st_p, r_p) dt, \\ Y &\sim \sum_{p=1}^N \sum_{k_1, \dots, k_p=0}^{N-p} \frac{(-1)^{|k|_1+p+1}}{|k|_1+p} \nabla^{k_p} (X \nabla^{k_{p-1}} (\cdots X \nabla^{k_1}(X) \cdots)) D^{-2(|k|_1+p)} \mod OP^{-N-1}. \end{aligned}$$

2. Action spectrale sur le tore non commutatif

Nous avons appliqué ces résultats au tore non commutatif. Il s'agit du triplet spectral non commutatif le plus simple possible. Il est basé sur l'algèbre $C^\infty(\mathcal{T}_\Theta^n)$ représentée par des fonctions rapidement décroissantes du type $\sum_{l \in \mathbb{Z}^n} a_l U_l$ où $(a_l) \in \mathcal{S}(\mathbb{Z}^n)$ et les éléments U_l sont des unitaires vérifiant la loi de commutation

$$U_l U_k = e^{-i l \cdot \Theta k} U_k U_l$$

où Θ est une matrice antisymétrique de déformation. L'opérateur de Dirac est de la forme $\mathcal{D} = -i\gamma^\mu \delta_\mu$, où les γ^μ sont les matrices gamma usuelles agissant sur $\mathbb{C}^{2^{[n/2]}}$ et $\delta_\mu(U_k) := ik_\mu U_k$. La fonctionnelle $\tau(a) := a_0$ où $a := \sum_l a_l U_l$ génère un espace de Hilbert \mathcal{H} de type GNS à partir duquel un triplet spectral réel régulier de dimension n peut être construit. Le calcul des intégrales non commutatives précédentes fait intervenir des termes du type JAJ^{-1} . Il en résulte que nous sommes alors amenés à étudier des résidus de séries de fonctions zêta pondérées par des suites rapidement décroissantes et faisant intervenir une phase dépendante de la pondération. Plus précisément, nous avons à étudier du point de vue de l'analyse complexe, les fonctions du type

$$g(s) := \sum_{l \in (\mathbb{Z}^n)^q} b(l) f_{\Theta \sum_{i=1}^q \varepsilon_i l_i}(s)$$

où $\varepsilon_i \in \{-1, 1\}$, $b \in \mathcal{S}((\mathbb{Z}^n)^q)$, $f_a(s) := \sum'_{k \in \mathbb{Z}^n} \frac{P(k)}{|k|^s} e^{2\pi i k \cdot a}$, $a \in \mathbb{R}^n$ et P un polynôme homogène de degré d . Il apparaît alors que nous pouvons connaître précisément les pôles de ces fonctions et calculer précisément les résidus correspondants si nous faisons une hypothèse reliée à la théorie de l'approximation diophantienne sur la matrice de déformation Θ . Plus exactement, on établit que si $\frac{1}{2\pi}\Theta$ est une matrice diophantienne, la fonction précédente g est méromorphe sur \mathbb{C} , avec au plus un pôle simple en $s = d + n$. Le résidu en ce pôle est (Theorem 2.4.4)

$$\text{Res}_{s=d+n} g(s) = \left(\sum_{l \in \mathcal{Z}} b(l) \right) \int_{u \in S^{n-1}} P(u) dS(u) = \sum_{l \in (\mathbb{Z}^n)^q} b(l) \text{Res}_{s=d+n} f_{\Theta \sum_{i=1}^q \varepsilon_i l_i}(s)$$

où $\mathcal{Z} := \{l \in (\mathbb{Z}^n)^q : \sum_{i=1}^q \varepsilon_i l_i = 0\}$. Finalement, nous obtenons grâce à ces résultats, le théorème suivant:

Théorème. *Si $\frac{1}{2\pi}\Theta$ est une matrice réelle antisymétrique diophantienne, alors le tore non commutatif (avec structure réelle) est un triplet spectral simple et son action spectrale est:*

(i) pour $n = 2$,

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = 4\pi \Phi_2 \Lambda^2 + \mathcal{O}(\Lambda^{-2}),$$

(ii) pour $n = 4$,

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = 8\pi^2 \Phi_4 \Lambda^4 - \frac{4\pi^2}{3} \Phi(0) \tau(F_{\mu\nu} F^{\mu\nu}) + \mathcal{O}(\Lambda^{-2}),$$

(iii) de façon générale:

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = \sum_{k=0}^n \Phi_{n-k} c_{n-k}(A) \Lambda^{n-k} + \mathcal{O}(\Lambda^{-1}),$$

où $c_{n-2}(A) = 0$, $c_{n-k}(A) = 0$ pour k impair. En particulier, $c_0(A) = 0$ quand n est impair.

Lorsque $n = 4$, un terme de type Yang-Mills non commutatif apparait au niveau du terme invariant d'échelle (en Λ^0): $\tau(F_{\mu\nu}F^{\mu\nu})$, où $F_{\mu\nu} := \delta_\mu(A_\nu) - \delta_\nu(A_\mu) - [A_\mu, A_\nu]$. La forme de l'action spectrale du tore non commutatif est donc très fortement similaire à celle du tore commutatif, à condition que la déformation Θ vérifie une condition diophantienne. Nous ne savons pas cependant si cette condition, bien que suffisante, est effectivement nécessaire pour obtenir une telle action. Nous conjecturons que c'est bien le cas, comme le suggère un argument heuristique (Remark 2.4.7).

3. Action spectrale sur $SU_q(2)$

Le troisième chapitre correspond à un travail en collaboration avec Bruno Iochum et Andrzej Sitarz: *Spectral action on $SU_q(2)$* [82].

L'objectif consiste à appliquer les techniques pseudodifférentielles vues précédemment au calcul de l'action spectrale sur le triplet spectral de Dąbrowski et al. [48] basé sur le groupe quantique $SU_q(2)$. Ce groupe quantique (ou algèbre de Hopf) peut être vu comme une q -déformation de la 3-sphère commutative.

L'algèbre $\mathcal{A} := \mathcal{A}(SU_q(2))$ de ce triplet est définie comme étant l'algèbre polynomialement engendrée par deux éléments a et b qui sont assujettis aux règles de commutation suivantes, où $0 < q < 1$:

$$ba = qab, \quad b^*a = qab^*, \quad bb^* = b^*b, \quad a^*a + q^2b^*b = 1, \quad aa^* + bb^* = 1.$$

On définit un espace de Hilbert $\mathcal{H} = \mathcal{H}^\uparrow \oplus \mathcal{H}^\downarrow$ avec les bases orthonormales $v_{m,l}^{j\uparrow}$ et $v_{m,l}^{j\downarrow}$ où $j \in \frac{1}{2}\mathbb{N}$, $0 \leq m \leq 2j$, $0 \leq l \leq 2j+1$ et $v_{m,l}^{j\downarrow}$ est nul si $j = 0$ ou $l = 2j$ or $2j+1$.

On représente alors \mathcal{A} avec l'application π initialement définie sur a, b et qui est donnée à la section 2.2. Cette représentation faisant intervenir des coefficients assez compliqués $\alpha_{j\mu n}^\pm, \beta_{j\mu n}^\pm$, et non diagonaux, il apparait très utile de définir une représentation approximée $\underline{\pi}$ telle que

$$\underline{\pi}(a) := a_+ + a_-, \quad \underline{\pi}(b) := b_+ + b_-$$

où on a posé (ici $q_n := \sqrt{1 - q^{2n}}$):

$$\begin{aligned} a_+ v_{m,l}^j &= q_{m+1} q_{l+1} v_{m+1,l+1}^{j+}, & a_- v_{m,l}^j &= q^{m+l+1} v_{m,l}^{j-}, \\ b_+ v_{m,l}^j &= q^l q_{m+1} v_{m+1,l}^{j+}, & b_- v_{m,l}^j &= -q^m q_l v_{m,l-1}^{j-}. \end{aligned}$$

Cette approximation ne modifiera pas les calculs d'intégrale non commutative puisque pour tout $x \in \mathcal{A}$, $\pi(x) - \underline{\pi}(x) \in \mathcal{K}_q$ où \mathcal{K}_q est un idéal inclus dans les opérateurs de type $OP^{-\infty}$. Autrement dit, $\pi(x) - \underline{\pi}(x)$ est un opérateur pseudodifférentiel régularisant.

L'opérateur de Dirac est défini de la façon suivante: $\mathcal{D} v_{ml}^j = \begin{pmatrix} 2j+\frac{3}{2} & 0 \\ 0 & -2j-\frac{1}{2} \end{pmatrix} v_{ml}^j$. Il possède donc le même spectre que l'opérateur de Dirac associé à la structure spinorielle de la 3-sphère standard. Cependant, l'opérateur de Dirac sur $SU_q(2)$ possède une caractéristique très particulière qui n'existe pas sur la 3-sphère commutative: le signe de \mathcal{D} , noté $F := \mathcal{D}|\mathcal{D}|^{-1}$, commute modulo $OP^{-\infty}$ avec les éléments de l'algèbre \mathcal{A} . Ceci a pour conséquence que les 1-formes $a[\mathcal{D}, b]$ sont essentiellement équivalentes aux δ -1-formes $a[|\mathcal{D}|, b]$ dans les calculs d'intégrales non commutatives. On peut construire avec cet opérateur de Dirac un triplet spectral $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ régulier

et de spectre de dimension $\{1, 2, 3\}$ sur $SU_q(2)$. D'autre part, une structure réelle J peut être construite sur ce triplet, avec la relation $J\mathcal{D} = \mathcal{D}J$. Cette structure réelle est définie par

$$J v_{m,l}^{j\uparrow} = i^{2(m+l)-1} v_{2j-m, 2j+1-l}^{j\uparrow}, \quad J v_{m,l}^{j\downarrow} = i^{-2(m+l)+1} v_{2j-m, 2j-1-l}^{j\downarrow}.$$

En utilisant une décomposition de type Poincaré–Birkhoff–Witt sur les δ -1-formes pour calculer les intégrales $\oint |D_A|^{-k}$, nous avons été amenés à étudier certains produits de séries faisant intervenir une inversion d'indice $m \mapsto 2j - m$ (sections 2.4.4 et 2.8.C, 2.8.D). Ces résultats ont permis de montrer que la structure réelle ne modifie pas le spectre de dimension $\{1, 2, 3\}$ de $SU_q(2)$, et que l'action spectrale du triplet $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ qui tient compte de la structure réelle, c'est à dire associée à la perturbation $\mathcal{D} \rightarrow \mathcal{D} + A + JAJ^{-1}$, est totalement déterminée par les intégrales suivantes (qui ne font plus intervenir la structure réelle J)

$$\oint A_\delta^q |\mathcal{D}|^{-p}, \quad 1 \leq q \leq p \leq 3,$$

où A_δ est une δ -1-forme. Afin de calculer précisément ces intégrales, un calcul différentiel sur $SU_q(2)$ modulo un idéal \mathcal{R} a été défini. Cet idéal est conçu de telle sorte que tout opérateur T dans \mathcal{R} est invisible par intégration non commutative avec $|\mathcal{D}|^{-2}, |\mathcal{D}|^{-3}$: $\oint T |\mathcal{D}|^{-2} = \oint T |\mathcal{D}|^{-3} = 0$. Le calcul d'intégrale du type $\oint A_\delta^q |\mathcal{D}|^{-p}$ est alors réduit à certains types particuliers de $\delta - 1$ -formes A_δ . Par exemple, il est possible d'obtenir toutes ces intégrales avec $p = 1$ à partir des intégrales suivantes (Proposition 3.5.16) :

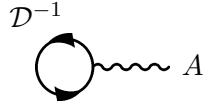
$$\begin{aligned} \oint (bb^*)^n |\mathcal{D}|^{-1} &= \frac{-2(1+q^{2n})}{(1-q^{2n})^2}, \\ \oint (bb^*)^n b^* \delta b |\mathcal{D}|^{-1} &= \oint (bb^*)^n b \delta b^* |\mathcal{D}|^{-1} = \frac{2}{1-q^{2n+2}}, \\ \oint (bb^*)^n a \delta a^* |\mathcal{D}|^{-1} &= \frac{-2q^{4n+2} - 2q^{4n} - 2q^{2n+2} + 6q^{2n}}{(1-q^{2n})^2(1-q^{2n+2})}, \\ \oint (bb^*)^n a^* \delta a |\mathcal{D}|^{-1} &= \frac{6q^{2n+2} - 2q^{2n} - 2q^2 - 2}{(1-q^{2n})^2(1-q^{2n+2})}. \end{aligned}$$

Ces résultats permettent finalement de retrouver toutes les actions spectrales possibles sur $SU_q(2)$. Nous avons constaté, à l'aide de certains exemples, que les termes de l'action spectrale n'ont pas toujours une limite finie lorsque $q \rightarrow 1$, c'est à dire lorsque $SU_q(2)$ "s'approche" de la 3-sphère commutative \mathbb{S}^3 . D'autre part, même lorsque cette limite existe, le terme obtenu n'est pas égal au terme correspondant à la 3-sphère. Il existe donc un "mur" entre la q -géométrie de $SU_q(2)$ et \mathbb{S}^3 qui n'apparaît pas au niveau des déformations du tore ou des plans de Moyal. Le calcul d'action spectrale sur d'autres géométries, telles que les sphères de Podleś [43, 46], le plan projectif [45], ou les q -sphères euclidiennes [44, 47, 96] pourrait faire l'objet d'investigations futures.

4. Tadpoles et triplets spectraux commutatifs

Ce chapitre présente un travail fait en collaboration avec Bruno Iochum: *Tadpoles and commutative spectral triples* [83]. Nous nous sommes intéressés à certaines questions concernant l'annulation d'intégrales non commutatives apparaissant dans l'action spectrale de Chamseddine–Connes. Plus particulièrement, nous avons étudié les intégrales du type $\oint A \mathcal{D}^{-1}$ (linéaires en A ,

où A est une 1-forme) qui correspondent, en théorie des champs, à des tadpoles. Ici, \mathcal{D}^{-1} est le propagateur fermionique de Feynman et $A\mathcal{D}^{-1}$ est un graphe à une boucle avec une ligne fermionique interne et une ligne bosonique externe :



Plus généralement, on définit un tadpole comme étant un terme linéaire en A apparaissant dans l'action spectrale $\mathcal{S}(\mathcal{D} + A, \Phi, \Lambda)$. Si d est la dimension d'un triplet spectral $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, on peut montrer que la partie linéaire en A du terme en Λ^{d-k} de l'action spectrale, que l'on note $\text{Tad}_{\mathcal{D}+A}(d-k)$ (tadpole d'ordre k) vérifie:

$$\begin{aligned} \text{Tad}_{\mathcal{D}+A}(d-k) &= -(d-k) \oint A \mathcal{D} |\mathcal{D}|^{-(d-k)-2}, \quad \forall k \neq d, \\ \text{Tad}_{\mathcal{D}+A}(0) &= - \oint A \mathcal{D}^{-1}. \end{aligned}$$

Il apparait alors que pour tout triplet spectral riemannien, c'est à dire du type $(\mathcal{A} := C^\infty(M), \mathcal{H} := L^2(M, S), \mathcal{D})$ où M est une variété compacte sans bord riemannienne à spin de dimension d , \mathcal{H} l'espace de Hilbert des spineurs de carré intégrable et \mathcal{D} l'opérateur de Dirac associé à la structure spin, tous ces termes sont nuls. Cette propriété est basée sur le fait que ce triplet est réel et commutatif. La structure réelle de $(\mathcal{A} := C^\infty(M), \mathcal{H} := L^2(M, S), \mathcal{D})$ provient de l'existence de la structure spin, car celle-ci implique l'existence d'un opérateur de conjugaison de charge J , qui est une isométrie antilinéaire satisfaisant:

$$JaJ^{-1} = a^*, \quad \forall a \in \mathcal{A}.$$

De façon plus générale, on peut montrer, en utilisant le résidu de Wodzicki, que $\oint B |\mathcal{D}|^{-(2k+1)}$, où B est un polynôme généré par \mathcal{A} et \mathcal{D} , est toujours nul si la dimension est paire, alors que $\oint B \mathcal{D}^{-2k}$ est toujours nul en dimension impaire. Dit autrement, $\oint B |\mathcal{D}|^{-(d-q)} = 0$ pour tout q impair.

Nous nous sommes aussi intéressés au cas d'une variété de dimension paire compacte avec bord, et d'une condition au bord de type chirale, c'est-à-dire telle que l'opérateur de Dirac perturbé $\mathcal{D} + A$ agit sur le domaine $\{s \in C^\infty(V) : \Pi_- s|_{\partial M} = 0\}$ où $\Pi_\pm := \frac{1}{2}(1 \pm \chi)$. Ici χ est un opérateur de chiralité sur le bord, c'est-à-dire tel que $\chi^2 = 1$ et

$$\{\chi, \gamma^d\} = 0, \quad [\chi, \gamma^a] = 0, \quad \forall a \in \{1, \dots, d-1\}.$$

On obtient alors une condition au bord mixte naturelle sur l'opérateur de type Laplace $(\mathcal{D} + A)^2$:

$$\mathcal{B}_\chi^A s := \Pi_- (D + A)^2 s|_{\partial M} \oplus \Pi_- s|_{\partial M}.$$

En se basant sur les formes explicites des coefficients du noyau de la chaleur dans le cadre des conditions aux bords mixtes [11, 12], on peut alors montrer qu'aucun tadpole ne peut exister dans l'action $\text{Tr}(\Phi((D + A)_{B_\chi^A}^2/\Lambda^2))$, au moins jusqu'à l'ordre 5. Nous nous attendons à ce que ceci se généralise à tous les ordres, et à d'autres types de conditions aux bords, notamment celles d'Atiyah-Patodi-Singer. Une approche de cette question, associée aux triplets spectraux commutatifs, fait l'objet d'un travail en cours [84].

5. Calcul pseudodifférentiel global sur variétés avec linéarisation

Le cinquième et dernier chapitre présente la construction d'un calcul symbolique et pseudodifférentiel global sur les variétés avec linéarisation.

Il a été montré par Gayral et al. [59] que les plans de Moyal sont des triplets spectraux non compacts. En d'autres termes, les plans de Moyal peuvent être vus comme des variétés à spin non compactes et non commutatives. Ce lien entre la théorie de la quantification par déformation et la géométrie non commutative montre en particulier que le paradigme des triplets spectraux est adapté aux questions de quantification. Le produit de Moyal est défini sur l'espace de Schwartz $\mathcal{S}(\mathbb{R}^{2n})$ des fonctions rapidement décroissantes

$$f \star g(x) := (\pi\theta)^{-2n} \int_{\mathbb{R}^{4n}} dy dz f(y) g(z) e^{\frac{2i}{\theta}(x-y) \cdot S(x-z)}$$

où $\theta \in \mathbb{R}^*$ et $S = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$, et donne à $\mathcal{S}(\mathbb{R}^{2n})$ une structure de pré- C^* -algèbre de Fréchet. Le triplet spectral décrit dans [59] est basé sur cette algèbre. L'extension de cette construction remarquable à un fibré cotangent T^*M d'une variété M non isométrique à \mathbb{R}^n reste un problème ouvert. Nous proposons dans ce chapitre la construction d'un calcul pseudodifférentiel global afin d'obtenir un produit de Moyal plus général que celui défini sur $\mathcal{S}(\mathbb{R}^{2n})$.

Le calcul pseudodifférentiel global [9, 49, 148, 149] permet d'établir une notion globale de symbole total d'un opérateur pseudodifférentiel, modulo l'algèbre résiduelle des opérateurs régularisant $\Psi^{-\infty}$ (à noyau lisse). Il est basé sur la définition d'une connexion sur la variété (ou plus généralement, d'une linéarisation [9]), et utilise l'application exponentielle, ainsi que le transport parallèle sur les géodésiques associées, pour obtenir un isomorphisme global (modulo $\Psi^{-\infty}$) entre les algèbres symboliques et opératorielles.

Lorsque la variété M n'est pas compacte, il apparaît utile, afin d'avoir une continuité de type $L^2(M)$ pour les opérateurs d'ordre 0, de considérer des espaces de symboles qui contrôlent à la fois la variable x et la covariable θ . Ces contrôles ont été utilisés sur \mathbb{R}^n dans le cadre du calcul pseudodifférentiel de type SG (voir par exemple [124]). Nous avons été amenés à définir un tel calcul dans le cadre des variétés à linéarisation, c'est à dire telles qu'il existe une application exponentielle abstraite (ou linéarisation) $\exp : TM \rightarrow M$ établissant un difféomorphisme $\exp_x : T_x M \rightarrow M$ en chaque point $x \in M$. Ce cadre est suffisamment général pour contenir les variétés de Cartan–Hadamard, qui sont les variétés simplement connexes, complètes, et de courbure négative.

L'outil essentiel de la définition du calcul global dans le cadre des variétés à linéarisation est la combinatoire liée à la formule de Faa-di-Bruno à plusieurs variables [39]. Celle-ci s'exprime de la façon suivante: si $f \in C^\infty(\mathbb{R}^p)$ et $g \in C^\infty(\mathbb{R}^n, \mathbb{R}^p)$ alors pour tout n -multi-indice $\nu \neq 0$,

$$\partial^\nu(f \circ g) = \sum_{1 \leq |\lambda| \leq |\nu|} (\partial^\lambda f) \circ g \sum_{s=1}^{|\nu|} \sum_{p_s(\nu, \lambda)} \nu! \prod_{j=1}^s \frac{1}{k^j!(l^j!)^{|k^j|}} (\partial^{l^j} g)^{k^j}$$

où les multi-indices k^j, l^j appartenant à l'ensemble $p_s(\nu, \lambda)$ vérifient $\sum_{j=1}^s k^j = \lambda$ et $\sum_{j=1}^s |k^j| l^j = \nu$. À l'aide de cette formule, il est possible de définir, de façon intrinsèque et dans le cadre des variétés à linéarisation, des espaces de Fréchet nucléaires de fonctions rapidement décroissantes $\mathcal{S}(M)$, $\mathcal{S}(M \times M)$, $\mathcal{S}(TM)$, $\mathcal{S}(T^*M)$ pourvu que l'application exponentielle satisfasse une hypothèse de contrôle à l'infini de type polynomial. Cette hypothèse dit plus précisément que les

applications de changement de coordonnées normales $\psi_{z,z'}^{\mathbf{b},\mathbf{b}'} := L_{\mathbf{b}} \exp_z^{-1} \exp_{z'} L_{\mathbf{b}'}^{-1}$ (où $z, z' \in M$, \mathbf{b}, \mathbf{b}' bases de $T_z M, T_{z'} M$ et $L_{\mathbf{b}}$ isomorphisme linéaire associé à \mathbf{b}) sont dans l'espace $\mathcal{O}_M(\mathbb{R}^n, \mathbb{R}^n)$ des fonctions (avec leurs dérivées) polynomialement dominées à l'infini.

Il est d'autre part possible de définir, sous ces conditions, des isomorphismes topologiques de quantifications \mathfrak{Op}_λ paramétrés par $\lambda \in [0, 1]$ qui permettent de passer de l'espace des opérateurs (plus exactement des noyaux) $\mathcal{S}'(M \times M)$ à l'espace des "symboles" $\mathcal{S}'(T^*M)$. Ces isomorphismes envoient $\mathcal{S}(M \times M)$ dans $\mathcal{S}(T^*M)$ et il apparait alors possible de définir un λ -produit (ou produit de Moyal si $\lambda = \frac{1}{2}$) sur $\mathcal{S}(T^*M)$ simplement par transfert de la convolution de noyau dans $\mathcal{S}(M \times M)$. Plus précisément, le produit

$$a \circ_\lambda b(x, \eta) = \int_{T_x(M) \times M} d\mu_x(\xi) d\mu(y) \int_{V_{x,\xi,y}^\lambda} d\mu_{x,\xi,y}^*(\theta, \theta') g_{x,\xi,y}^\lambda e^{2\pi i \omega_{x,\xi,y}^\lambda(\eta, \theta, \theta')} a(y_{x,\xi}^\lambda, \theta) b(y_{x,-\xi}^{1-\lambda}, \theta')$$

où les applications V, g, ω sont déterminées (Proposition 5.3.11) par l'application exponentielle et la densité $d\mu$ considérée sur la variété, donne à $\mathcal{S}(T^*M)$ une structure d'algèbre de Fréchet, et se réduit précisément au produit de Moyal classique lorsque $\lambda = \frac{1}{2}$ et $M = \mathbb{R}^n$.

Nous avons ensuite étudié l'extension du SG -calcul sur les variétés à linéarisation. Les espaces de symboles $\mathcal{S}^{l,m}$ (ayant un contrôle séparé en x et θ) peuvent être définis si on renforce l'hypothèse précédente de contrôle polynomial sur les difféomorphismes $\psi_{z,z'}^{\mathbf{b},\mathbf{b}'}$. Plus précisément, si les applications $\psi_{z,z'}^{\mathbf{b},\mathbf{b}'}$ vérifient (S_1) , c'est-à-dire si pour tout n -multi-indice $\alpha \neq 0$ (ici $\langle x \rangle := (1 + \|x\|^2)^{1/2}$)

$$\partial^\alpha \psi_{z,z'}^{\mathbf{b},\mathbf{b}'}(x) = \mathcal{O}(\langle x \rangle^{1-|\alpha|}),$$

alors les espaces $\mathcal{S}^{l,m}$ de symboles a , définis par les estimations suivantes, pour tout système de coordonnées normal (z, \mathbf{b}) ,

$$\partial_{z,\mathbf{b}}^{(\alpha,\beta)} a(x, \theta) = \mathcal{O}(\langle x \rangle_{z,\mathbf{b}}^{l-|\alpha|} \langle \theta \rangle_{z,\mathbf{b},x}^{m-|\beta|}),$$

sont des espaces de Fréchet homéomorphes aux espaces de SG -symboles classiques sur \mathbb{R}^n . A partir de ces espaces, on peut définir les opérateurs pseudodifférentiels sur M en posant $\Psi^{l,m} := \mathfrak{Op}_\lambda(\mathcal{S}^{l,m})$ pourvu que cet espace ne dépende pas du paramètre de quantification λ et qu'il stabilise à la fois $\mathcal{S}(M)$ et $\mathcal{S}'(M)$. Ceci a été rendu possible par une analyse des espaces d'amplitudes a et des opérateurs associés

$$\langle \mathfrak{Op}_\Gamma(a), u \rangle := \int_{\mathbb{R}^{3n}} e^{2\pi i \langle \vartheta, \zeta \rangle} \text{Tr} (a(x, \zeta, \vartheta) \Gamma(u)^*(x, \zeta)) d\zeta d\vartheta dx$$

où Γ est un isomorphisme topologique de $\mathcal{S}(\mathbb{R}^{2n})$ vérifiant certaines hypothèses de contrôle à l'infini (Proposition 5.4.14, 5.4.17 et Lemma 5.4.18).

La partie suivante est consacrée à un résultat sur la composition des opérateurs pseudodifférentiels. Il est établi (Theorem 5.4.47) sous une hypothèse particulière (C_σ) (Definition 5.4.37) sur l'application exponentielle, que $\Psi := \cup_{l,m} \Psi^{l,m}$ est une $*$ -algèbre sous la composition d'opérateurs et le symbole (pour $\lambda = 0$) du produit de deux opérateurs A, B satisfait la relation asymptotique

$$\sigma_0(AB)_{z,\mathbf{b}} \sim \sum_{\beta, \gamma \in \mathbb{N}^n} c_{\beta\gamma} \partial_{\zeta,\vartheta}^{\gamma,\gamma} (a(x, \vartheta) \partial_{\zeta'}^\beta (e^{2\pi i \langle \vartheta, \varphi_{x,\zeta}(\zeta') \rangle} (\partial_{\vartheta'}^\beta f_b)(x, \zeta, \zeta', L_{x,\zeta}(\vartheta))))_{\zeta'=0} \Big|_{\zeta=0}$$

où $a := \sigma_0(A)_{z,\mathfrak{b}}$, $b := \sigma_0(B)_{z,\mathfrak{b}}$, et les termes L, ϕ capturent la "courbure" liée à l'application exponentielle abstraite \exp . Nous montrons en dernière partie que l'espace hyperbolique \mathbb{H} de dimension 2 est une variété avec linéarisation (l'application exponentielle étant celle venant de la structure riemannienne) vérifiant l'hypothèse de contrôle (S1). Ceci permet de définir de façon globale et intrinsèque les espaces de Schwartz $\mathcal{S}(\mathbb{H})$, $\mathcal{S}(T^*\mathbb{H})$, $\mathcal{S}(T\mathbb{H})$ ainsi que les espaces de symboles $S_1^{l,m}(T^*\mathbb{H})$.

L'analyse détaillée des λ -produits sur $\mathcal{S}(T^*\mathbb{H})$ (ou plus généralement sur $\mathcal{S}(T^*M)$, pour M avec \mathcal{O}_M -linéarisation) et les propriétés spectrales associées restent à étudier. Il serait par exemple intéressant de voir sous quelle condition les algèbres $(\mathcal{S}(T^*M), \circ_\lambda)$ peuvent générer un triplet spectral non compact. D'autre part, il est possible d'envisager de connecter ou d'étendre le calcul symbolique présenté ici avec les opérateurs de Fourier intégraux [42, 117, 118], les traces régularisées [110] et les espaces de Gelfand–Shilov [15].

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